Hawking-Unruh Radiation and Radiation of a Uniformly Accelerated Charge

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Abstract

Hawking-Unruh radiation is a measure of the quantum fluctuations in the radiation of accelerated charges. In the case of uniform acceleration, the existence of Unruh radiation adds confidence that radiation exists at all, and helps clarify aspects of the equivalence between radiation in uniform acceleration and in a uniform gravitational field.

1 Hawking-Unruh Radiation

According to Hawking [1], an observer outside a black hole experiences a bath of thermal radiation of temperature

$$T = \frac{\hbar g}{2\pi c k} , \tag{1}$$

where g is the local acceleration due to gravity, c is the speed of light, $\hbar$ is Planck's constant and k is Boltzmann's constant. In some manner, the background gravitational field interacts with the quantum fluctuations of the electromagnetic field with the result that energy can be transferred to the observer as if he(she) were in an oven filled with black-body radiation. Of course, the effect is strong only if the background field is strong.

An extreme example is that if the temperature is equivalent to 1 MeV or more, virtual electron-positron pairs emerge from the vacuum into real particles.

As remarked by Unruh [2], this phenomenon can be demonstrated in the laboratory according to the principle of equivalence: an accelerated observer in a gravity-free environment experiences the same physics (locally) as an observer at rest in a gravitational field. Therefore, an accelerated observer (in zero gravity) should find him(her)self in a thermal bath of radiation characterized by temperature

$$T = \frac{\hbar a^\star}{2\pi c k} , \tag{2}$$

where $a^\star$ is the acceleration as measured in the observer's instantaneous rest frame.

This topic has been reviewed in the Journal in Ref. [3].

The Hawking-Unruh temperature finds application in accelerator physics as the reason that electrons in a storage ring do not reach 100% polarization despite emitting polarized synchrotron radiation [4]. Indeed, the various limiting features of performance of a storage ring that arise due to quantum fluctuations of the synchrotron radiation [5] can be understood quickly in terms of eq. (2) [6, 7].

Here we consider another aspect of the Hawking-Unruh effect. Suppose the observer is an electron accelerated by an electromagnetic field E. Then, scattering of the electron off photons in the apparent thermal bath would be interpreted by a laboratory observer as an "extra" contribution to the radiation rate of the accelerated charge [8]. The power of the "extra" radiation, which I call "Unruh radiation", is given by

$$\begin{aligned}\frac{dU_{\rm Unruh}}{dt} &= \text{(energy flux of thermal radiation)} \\ &\quad \times\text{(scattering cross section)}.\end{aligned} \tag{3}$$

For the scattering cross section, we use the well-known result for Thomson scattering, $\sigma_{\rm Thomson} = 8\pi r_0^2/3$, where $r_0 = e^2/mc^2$ is the classical electron radius and m is the mass of the electron. The energy density of thermal radiation is given by the usual expression of Planck:

$$\frac{dU}{d\nu} = \frac{8\pi}{c^3}\frac{h\nu^3}{e^{h\nu/kT}-1}, \tag{4}$$

where ν is the frequency. The flux of the isotropic radiation on the electron is just c times the energy density. Note that these relations hold in the instantaneous rest frame of the electron. Then

$$\frac{dU_{\rm Unruh}}{dtd\nu} = \frac{8\pi}{c^2}\frac{h\nu^3}{e^{h\nu/kT}-1}\frac{8\pi}{3}r_0^2. \tag{5}$$

On integrating over ν we find

$$\frac{dU_{\rm Unruh}}{dt} = \frac{8\pi^3\hbar r_0^2}{45c^2}\left(\frac{kT}{\hbar}\right)^4 = \frac{\hbar r_0^2 a^{\star 4}}{90\pi c^6}, \tag{6}$$

using the Hawking-Unruh relation (2). The presence of $\hbar$ in eq. (6) reminds us that Unruh radiation is a quantum effect.

This equals the classical Larmor radiation rate, $dU/dt = 2e^2 a^{\star 2}/3c^3$, when

$$E^\star = \sqrt{\frac{60\pi}{\alpha}}E_{\rm crit} \approx \frac{E_{\rm crit}}{\alpha}, \tag{7}$$

where $E_{\rm crit}$ is the quantum electrodynamic critical field strength,

$$E_{\rm crit} = \frac{m^2c^3}{e\hbar} = 1.6\times 10^{16}\ \text{V/cm} = 3.3\times 10^{13}\ \text{Gauss}, \tag{8}$$

that was first noted in the context of Klein's paradox [9, 10, 11]. In this case, the acceleration $a^\star = eE^\star/m$ is about 10^{31} Earth g's.

The "Unruh radiation" deduced above is, however, not really a new type of radiation. Sciama [12] has emphasized how the apparent temperature of an accelerated observed should be interpreted in view of quantum fluctuations. Unruh radiation is a quantum correction to the classical radiation rate that grows large only in situations where quantum fluctuations in the radiation rate become very significant. This phenomenon should be contained in the standard theory of quantum electrodynamics, but a direct demonstration of this is not yet available.

2 Radiation During Uniformly Accelerated Motion

The existence of Unruh radiation provides an interesting comment on the "perpetual problem" of whether a uniformly accelerated charge emits electromagnetic radiation [13]. This famous problem arises in discussions of the radiation reaction that began with Lorentz [14], and Planck [15, 16]. The (nonrelativistic) equation of motion including the radiation reaction is (in Gaussian units)

$$m\dot{\mathbf{v}} = \mathbf{F}_{\rm ext} + \mathbf{F}_{\rm react}, \tag{9}$$

where $\mathbf{F}_{\rm ext}$ is an external force on the electron due either to an electromagnetic or gravitational field,

$$\mathbf{F}_{\rm react} = \frac{2e^2}{3c^3}\ddot{\mathbf{v}} + \mathcal{O}(\mathbf{v}/c) \tag{10}$$

is the radiation reaction force, $\mathbf{v}$ is the velocity of the electron and the dot indicates differentiation with respect to time.

The "perpetual problem" is whether a charge radiates if its acceleration is uniform, *i.e.*, if $\ddot{\mathbf{v}} = 0$. In this case the radiation reaction force (10) vanishes and many people have argued that this means there is no radiation [17, 18, 19, 20].

An additional perspective on this problem comes with the use of covariant notation.

The relativistic version of (9) in 4-vector notation is

$$mc^2\frac{du^\mu}{ds} = F^\mu_{\rm ext} + F^\mu_{\rm react}, \tag{11}$$

with external 4-force $F^\mu_{\rm ext}$, and radiation-reaction 4-force given by

$$F^\mu_{\rm react} = \frac{2e^2}{3}\frac{d^2u^\mu}{ds^2} - \frac{Ru^\mu}{c}, \tag{12}$$

where

$$R = -\frac{2e^2c}{3}\frac{du_\nu}{ds}\frac{du^\nu}{ds} = \frac{2e^2\gamma^6}{3c^3}\left[\dot{\mathbf{v}}^2 - \frac{(\mathbf{v}\times\dot{\mathbf{v}})^2}{c^2}\right] \geq 0 \tag{13}$$

is the invariant rate of radiation of energy of an accelerated charge, $u^\mu = \gamma(1, \mathbf{v}/c)$ is the 4-velocity, $\gamma = 1/\sqrt{1 - v^2/c^2}$, $ds = cd\tau$ is the invariant interval and the metric is $(1, -1, -1, -1)$.

The time component of eq. (11) can be written

$$\frac{d\gamma mc^2}{dt} = \mathbf{F}_{\rm ext}\cdot\mathbf{v} + \frac{dQ}{dt} - R, \tag{14}$$

where

$$Q = \frac{2\gamma^4 e^2\mathbf{v}\cdot\dot{\mathbf{v}}}{3c^3}, \tag{15}$$

is an energy first identified by Schott [21, 22] as being stored "in the electron in virtue of its acceleration", and which was given the name "acceleration energy" by him. The space components are

$$\frac{d\gamma m\mathbf{v}}{dt} = \mathbf{F}_{\rm ext} + \frac{2e^2\gamma^2}{3c^3} \tag{16}$$
$$\times \left[\ddot{\mathbf{v}} + \frac{3\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})\dot{\mathbf{v}} + \frac{\gamma^2}{c^2}(\mathbf{v}\cdot\ddot{\mathbf{v}})\mathbf{v} + \frac{3\gamma^4}{c^4}(\mathbf{v}\cdot\dot{\mathbf{v}})^2\mathbf{v}\right].$$

Equations equivalent to (14-16) were first given by Abraham [23]. Von Laue [24] was the first to show that these equations can be obtained by a Lorentz transformation of the nonrelativistic results (9-10). The covariant notation of eqs. (11-13) was first applied to the radiation reaction by Dirac [25]. An interesting discussion of the development of eqs. (14-16) has been given recently by Yaghjian [26].

In the case of uniform acceleration, Schott [22] argued that "the energy radiated by the electron is derived entirely from its acceleration energy; there is as it were internal compensation amongst the different parts of its radiation pressure, which causes its resultant effect to vanish". This view is somewhat easier to follow if "acceleration energy" means energy stored in the near and induction zones of the electromagnetic field, as argued by Thirring [28] and by Fulton and Rohrlich [31]. Other commentary on this problem includes that of Drukey [27], Bondi and Hoyle [29], DeWitt[30, 36], Rohrlich[32, 37], Rosen [33], Bradbury [34], Leibovitz [35], Nikishov [38], Ginzburg [13], Herrera [39] and Coleman [40], all of whom concur that a uniformly accelerated charge radiates.

3 Implications of Hawking-Unruh Radiation

An immediate consequence of the existence of Unruh radiation is that its interpretation as a measure of the quantum fluctuations about the rate of classical radiation implies that the classical radiation exists.

At present, Unruh radiation for uniformly accelerated motion exists only as a theoretical concept, not yet confirmed in the laboratory. Experimental evidence for Hawking-Unruh effects does exist for uniform circular motion, as mentioned in the Introduction.

It is noteworthy that while discussion of radiation by an accelerated charge is perhaps most intricate classically in case of uniform acceleration, the discussion of quantum fluctuations is the most straightforward for uniform acceleration.

In addition, Hawking-Unruh radiation helps clarify a residual puzzle in the discussion of the equivalence between accelerated charges and charges in a gravitational field. Because of the difficulty in identifying an unambiguous wave zone for uniformly accelerated motion of a charge (in a gravity-free region) and also in the case of a charge in a uniform gravitational field, there remains some doubt as to whether the "radiation" deduced by classical arguments contains photons. Thus, on p. 573 of the article by Ginzburg [13] we read: "neither a homogeneous gravitational field nor a uniformly accelerated reference frame can actually "generate" free particles, expecially photons". We now see that the quantum view is richer than anticipated, and that Hawking-Unruh radiation provides at least a partial understanding of particle emission in uniform acceleration or gravitation. Hence, we can regard the concerns of Bondi and Gold [29], Fulton and Rohrlich [31], the DeWitt's [36] and Ginzburg [13] on radiation and the equivalence principle as precursors to the concept of Hawking radiation.

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