Gaussian Laser Beams with Radial Polarization

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(March 14, 2000; updated December 2, 2012)

1 Problem

Deduce a solution for a Gaussian laser beam in vacuum with radial polarization of the electric field.

The solution should be expressed in terms of three geometric parameters of a focused beam, the diffraction angle θ_0, the waist w_0, and the depth of focus (Rayleigh range) z_0, which are related by,

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \qquad \text{and} \qquad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}, \tag{1}$$

where the axis of the beam is the z axis, as shown in Fig. 1.

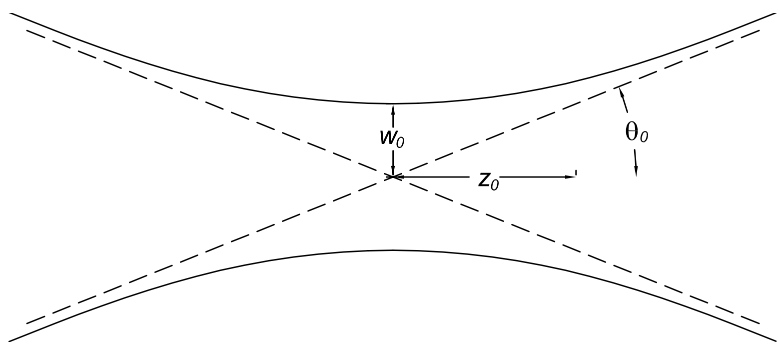

Figure 1: The geometry of a focused, cylindrical beam is expressed in terms of the three parameters w_0 (the "waist"), z_0 (the Rayleigh range = depth of focus) and θ_0 (the diffraction angle) that are related by eq. (1).

In general, the forms of laser beams can be usefully deduced from a vector potential that has a single Cartesian coordinate. Linearly polarized beams result from a vector potential with only A_x or A_y nonzero, while the radially polarized beam results from having only A_z nonzero.

2 Solution

If a laser beam is to have radial transverse polarization, the transverse electric must vanish on the symmetry axis, which is charge free in vacuum. However, we can expect a nonzero longitudinal electric field on the axis, noting that the projections onto the axis of the electric field vectors of rays all have the same sign, as shown in Fig. 2a. This contrasts with the case of linearly polarized Gaussian laser beams [2, 3, 4, 5] for which rays at 0° and 180° azimuth to the polarization direction have axial electric field components of opposite sign, as shown in Fig. 2b. The longitudinal electric field of radially polarized laser beams may be able to transfer net energy to charged particles that propagate along the optical axis, providing a form of laser acceleration [6, 7, 8, 9, 10].

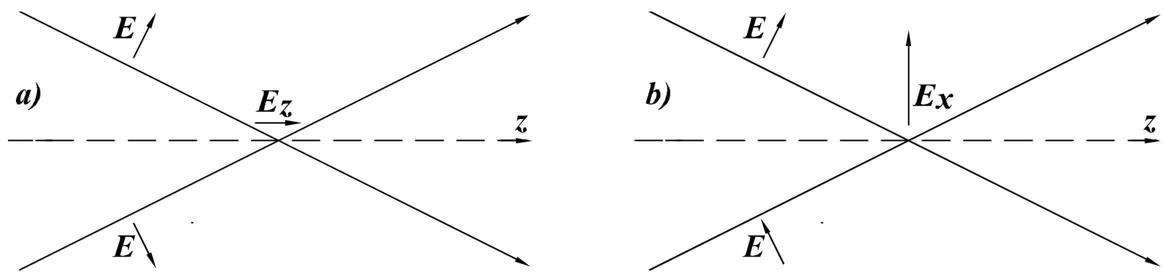

Figure 2: a) The radial polarization of the electric field leads to a longitudinal electric field at the focus. b) For a linearly polarized laser beam, shown here with polarization along the x axis, the electric field is transverse at the focus.

Although two of the earliest papers on Gaussian laser beams [11, 12] discuss radially polarized modes (without deducing the simplest mode), most subsequent literature has emphasized linearly polarized Gaussian beams. We demonstrate that a calculation that begins with the vector potential (sec. 2.1) leads to both the lowest-order linearly and radially polarized modes. We include a discussion of Gaussian laser pulses as well as continuous beams, and find in sec. 2.2 that the temporal pulse shape must obey condition (9). The paraxial wave equation and its lowest-order, linearly polarized solutions are reviewed in secs. 2.3-4. Readers familiar with the paraxial wave equation for linearly polarized Gaussian beams may wish to skip directly to sec. 2.5 in which the radially polarized mode is displayed.

2.1 Solution via the Vector Potential

Many discussions of Gaussian laser beams emphasize a single electric field component such as $E_x = f(r, z)e^{i(kz-\omega t)}$ of a cylindrically symmetric beam of angular frequency ω and wave number $k = \omega/c$ propagating in vacuum along the z axis. Of course, the electric field must satisfy the free-space Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$. If $f(r, z)$ is not constant and $E_y = 0$, then we must have nonzero E_z. That is, the desired electric field has more than one vector component.

To deduce all components of the electric and magnetic fields of a Gaussian laser beam from a single scalar wave function, we follow the suggestion of Davis [13] and seek solutions for a vector potential $\mathbf{A}$ that has only a single Cartesian component (such that $(\nabla^2\mathbf{A})_j = \nabla^2 A_j$ [14]). We work in the Lorenz gauge (and Gaussian units), so that the scalar potential Φ is related to the vector potential by,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \tag{2}$$

The vector potential can therefore have a nonzero divergence, which permits solutions having only a single component. Of course, the electric and magnetic fields can be deduced from the potentials via,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{3}$$

and,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{4}$$

For this, the scalar potential must first be deduced from the vector potential using the Lorenz condition (2).

The vector potential satisfies the free-space wave equation,

$$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{5}$$

We seek a solution in which the vector potential is described by a single Cartesian component A_j that propagates in the $+z$ direction with the form,

$$A_j(\mathbf{r}, t) = \psi(r_\perp, z) g(\varphi) e^{i\varphi}, \tag{6}$$

where the spatial envelope ψ is azimuthally symmetric, $r_\perp = \sqrt{x^2 + y^2}$, g is the temporal pulse shape, and the phase φ is given by,

$$\varphi = kz - \omega t. \tag{7}$$

Inserting trial solution (6) into the wave equation (5) we find that,

$$\nabla^2 \psi + 2ik \frac{\partial \psi}{\partial z}\left(1 - \frac{ig'}{g}\right) = 0, \tag{8}$$

where $g' = dg/d\varphi$.

2.2 A Condition on the Temporal Pulse Shape $g(\varphi)$

Since ψ is a function of $\mathbf{r}$ while g and g' are functions of the phase φ, eq. (8) cannot be satisfied in general. Often the discussion is restricted to the case where $g' = 0$, *i.e.*, to continuous waves. For a pulsed laser beam, g must obey,

$$\left|\frac{g'}{g}\right| \ll 1 \tag{9}$$

for eq. (8) to be consistent.

It is noteworthy that a "Gaussian" laser beam cannot have a Gaussian temporal pulse. That is, if $g = \exp[-(\varphi/\varphi_0)^2]$, then $|g'/g| = 2|\varphi|/\varphi_0^2$, which does not satisfy condition (9) for $|\varphi|$ large compared to the characteristic pulsewidth $\varphi_0 = \omega\Delta t$, *i.e.*, in the tails of the pulse.

A more appropriate form for a pulsed beam is a hyperbolic secant (as arises in studies of solitons),

$$g(\varphi) = \operatorname{sech}\left(\frac{\varphi}{\varphi_0}\right). \tag{10}$$

Then, $|g'/g| = (1/\varphi_0)|\tanh(\varphi/\varphi_0)|$, which is less than one everywhere provided that $\varphi_0 \gg 1$.

2.3 The Paraxial Wave Equation

In the remainder of this paper, we suppose that condition (9) is satisfied. Then, the differential equation (8) for the spatial envelope function ψ becomes,

$$\nabla^2\psi + 2ik\frac{\partial\psi}{\partial z} = 0. \tag{11}$$

The function ψ can and should be expressed in terms of three geometric parameters of a focused beam, the diffraction angle θ_0, the waist w_0, and the depth of focus (Rayleigh range) z_0, which are related by,

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \qquad \text{and} \qquad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}. \tag{12}$$

We therefore work in the scaled coordinates,

$$\xi = \frac{x}{w_0}, \qquad \upsilon = \frac{y}{w_0}, \qquad \rho^2 = \frac{r_\perp^2}{w_0^2} = \xi^2 + \upsilon^2, \qquad \text{and} \qquad \varsigma = \frac{z}{z_0}, \tag{13}$$

Changing variables and noting relations (12), eq. (11) takes the form,

$$\nabla_\perp^2\psi + 4i\frac{\partial\psi}{\partial\varsigma} + \theta_0^2\frac{\partial^2\psi}{\partial\varsigma^2} = 0, \tag{14}$$

where,

$$\nabla_\perp^2\psi = \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\upsilon^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right), \tag{15}$$

since ψ is independent of the azimuth ϕ.

The form of eq. (14) suggests the series expansion,

$$\psi = \psi_0 + \theta_0^2\psi_2 + \theta_0^4\psi_4 + ... \tag{16}$$

in terms of the small parameter θ_0^2. Inserting this into eq. (14) and collecting terms of order θ_0^0 and θ_0^2, we find,

$$\nabla_\perp^2\psi_0 + 4i\frac{\partial\psi_0}{\partial\varsigma} = 0, \tag{17}$$

and,

$$\nabla_\perp^2\psi_2 + 4i\frac{\partial\psi_2}{\partial\varsigma} = -\frac{\partial^2\psi_0}{\partial\varsigma^2}, \tag{18}$$

etc.

Equation (17) is called the the paraxial wave equation, whose solution we obtain by an "educated guess". Namely, we expect the transverse behavior of the wave function ψ_0 to be Gaussian, but with a width that varies with z. Also, the amplitude of the wave should vary with z, asypmtotically falling as $1/z$. We work in the scaled coordinates ρ and ς, and write a trial solution as,

$$\psi_0 = h(\varsigma)e^{-f(\varsigma)\rho^2}, \tag{19}$$

where the possibly complex functions f and h are defined to obey $f(0) = 1 = h(0)$. Since the transverse coordinate ρ is scaled by the waist w_0, we see that $Re(f) = w_0^2/w^2(\varsigma)$ where $w(\varsigma)$ is the beam width at position ς. From the geometric parameters (12) we see $w(\varsigma) \approx \theta_0 z = w_0\varsigma$ for large ς. Hence, we expect that $Re(f) \approx 1/\varsigma^2$ for large ς. Also, we expect the amplitude h to obey $|h| \approx 1/\varsigma$ for large ς.

Plugging the trial solution (19) into the paraxial wave equation (17) we find that,

$$-fh + ih' + \rho^2 h(f^2 - if') = 0. \tag{20}$$

For this to be true at all values of ρ we must have,

$$\frac{f'}{f^2} = -i, \qquad \text{and} \qquad \frac{h'}{fh} = -i. \tag{21}$$

We see that $f = h$ is a solution – despite the different physical origin of these two functions as the transverse width and amplitude of the wave. We integrate the first of eq. (21) to obtain,

$$\frac{1}{f} = C + i\varsigma. \tag{22}$$

Our definition $f(0) = 1$ determines that $C = 1$. That is,

$$f = \frac{1}{1 + i\varsigma} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}}. \tag{23}$$

Note that $Re(f) = 1/(1 + \varsigma^2) = w_0^2/w^2(\varsigma)$, while $|f| = 1/\sqrt{1 + \varsigma^2}$, so that $f = h$ is in fact consistent with the asymptotic expectations discussed above. The longitudinal dependence of the width of the Gaussian beam is now seen to be,

$$w(\varsigma) = w_0\sqrt{1 + \varsigma^2}. \tag{24}$$

The lowest-order wave function is,

$$\psi_0 = fe^{-f\rho^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}} e^{-\rho^2/(1+\varsigma^2)} e^{i\varsigma\rho^2/(1+\varsigma^2)}. \tag{25}$$

The factor $e^{-i\tan^{-1}\varsigma}$ in ψ_0 is the so-called Gouy phase shift, which changes from 0 to $\pi/2$ as z varies from 0 to ∞, with the most rapid change near the z_0. For large z the phase factor $e^{i\varsigma\rho^2/(1+\varsigma^2)}$ can be written as $e^{ikr_\perp^2/(2z)}$, recalling eq. (12). When this is combined with the travelling wave factor $e^{i(kz-\omega t)}$ we have,

$$e^{i[kz(1+r_\perp^2/2z^2)-\omega t]} \approx e^{i(kr-\omega t)}, \tag{26}$$

where $r = \sqrt{z^2 + r_\perp^2}$. Thus, the wave function ψ_0 is a modulated spherical wave for large z, but is a modulated plane wave near the focus.[1]

[1]For $\varsigma \gg 1$ and $\rho \ll \varsigma$ the wavefunction $\psi_0\, e^{i(kz-\omega t)}$ is approximatelty $e^{ikr'}/r'$ (to within a constant, complex factor), where $r'(x, y, z) = \sqrt{r_\perp^2 + (z - iz_0)^2}$ is the distance from the complex coordinate $(0, 0, iz_0)$ [15, 16]. This leads to the description of paraxial Gaussian beams as consisting of "complex rays."

The solution to eq. (18) for ψ_2 has been given in [13], and that for ψ_4 has been discussed in [17].

With the lowest-order spatial function ψ_0 in hand, we are nearly ready to display the electric and magnetic fields of the corresponding Gaussian beams. But first, we need the scalar potential Φ, which we suppose has the form,

$$\Phi(\mathbf{r},t) = \Phi(\mathbf{r})g(\varphi)e^{i\varphi}, \tag{27}$$

similar to that of the vector potential. Then,

$$\frac{\partial\Phi}{\partial t} = -i\omega\Phi\left(1 - \frac{ig'}{g}\right) \approx -i\omega\Phi, \tag{28}$$

assuming condition (9) to be satisfied. In that case,

$$\Phi = -\frac{i}{k}\boldsymbol{\nabla}\cdot\mathbf{A}, \tag{29}$$

according to the Lorentz condition (2). The electric field is then given by,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \approx ik\left[\mathbf{A} + \frac{1}{k^2}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})\right] = \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B}, \tag{30}$$

in view of condition (9). Note that $(1/k)\partial/\partial x = (\theta_0/2)\partial/\partial\xi$, *etc.*, according to eqs. (12)-(13).

2.4 Linearly Polarized Gaussian Beams

Taking the scalar wave function (25) to be the x component of the vector potential,

$$A_x = \frac{E_0}{ik}\psi_0 g(\varphi)e^{i\varphi}, \qquad A_y = A_z = 0, \tag{31}$$

the corresponding electric and magnetic fields are found from eqs. (4), (30) and (31) to be the familiar forms of a linearly polarized Gaussian beam,

$$\begin{aligned}
E_x &= E_0\psi_0 g e^{i\varphi} + \mathcal{O}(\theta_0^2) \approx E_0 f e^{-f\rho^2} g e^{i\varphi} \\
&= \frac{E_0 e^{-\rho^2/(1+\varsigma^2)}g(\varphi)}{\sqrt{1+\varsigma^2}} e^{i[kz+\varsigma\rho^2/(1+\varsigma^2)-\omega t-\tan^{-1}\varsigma]}, \\
&= \frac{E_0 e^{-r_\perp^2/w^2(z)}g(\varphi)}{\sqrt{1+z^2/z_0^2}} e^{i\{kz[1+r_\perp^2/2(z^2+z_0^2)]-\omega t-\tan^{-1}(z/z_0)\}}, \\
E_y &= 0, \\
E_z &= \frac{i\theta_0 E_0}{2}\frac{\partial\psi_0}{\partial\xi} g e^{i\varphi} + \mathcal{O}(\theta_0^3) \approx -i\theta_0 f\xi E_x,
\end{aligned} \tag{32}$$

$$\begin{aligned}
B_x &= 0, \\
B_y &= E_x, \\
B_z &= \frac{i\theta_0 E_0}{2}\frac{\partial\psi_0}{\partial\upsilon} g e^{i\varphi} = -i\theta_0 f\upsilon E_x,
\end{aligned} \tag{33}$$

where,

$$w(z) = w_0\sqrt{1 + z^2/z_0^2} \tag{34}$$

is the characteristic transverse size of the beam at position z. Near the focus ($r_\perp \lesssim w_0, |z| < z_0$), the beam is a plane wave,

$$E_x \approx E_0 e^{-r_\perp^2/w_0^2} e^{i(kz-\omega t-z/z_0)}, \qquad E_z \approx \theta_0 \frac{x}{w_0} E_0 e^{-r_\perp^2/w_0^2} e^{i(kz-\omega t-2z/z_0-\pi/2)}, \tag{35}$$

For large z,

$$E_x \approx E_0 e^{-\theta^2/\theta_0^2} \frac{e^{i(kr-\omega t-\pi/2)}}{r}, \qquad E_z \approx -\frac{x}{r} E_x, \tag{36}$$

where $r = \sqrt{r_\perp^2 + z^2}$ and $\theta \approx r_\perp/r$, which describes a linearly polarized spherical wave of extent θ_0 about the z axis. The fields E_x and E_z, *i.e.*, the real parts of eqs. (36), are shown in Figs. 3 and 4.

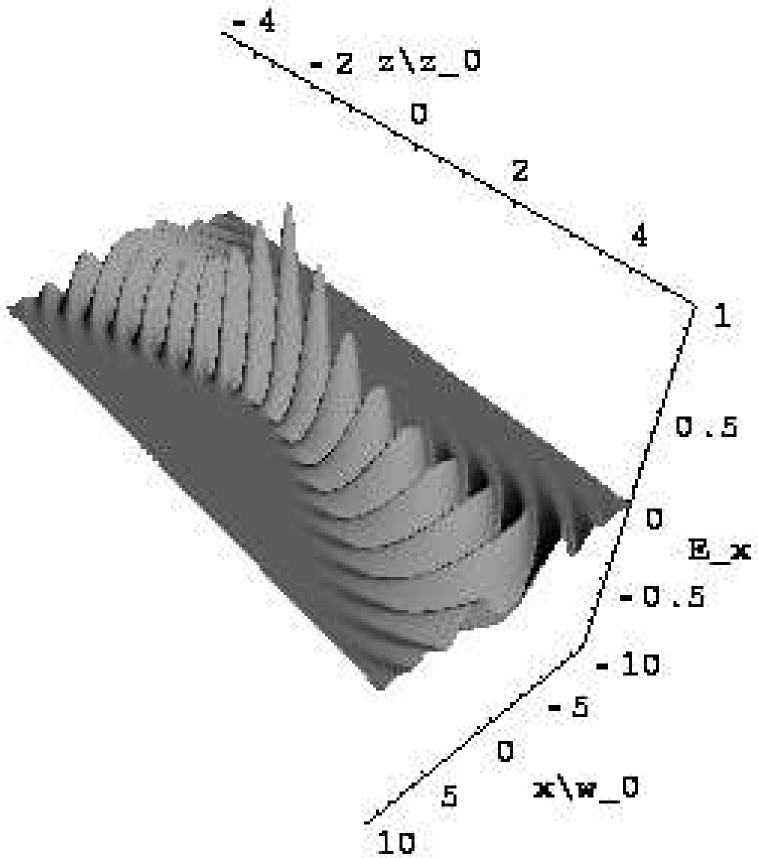

Figure 3: The electric field $E_x(x, 0, z)$ of a linearly polarized Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (34).

The fields (32)-(33) satisfy $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 = \boldsymbol{\nabla}\cdot\mathbf{B}$ plus terms of order θ_0^2.

Clearly, a vector potential with only a y component of form similar to eq. (31) leads to the lowest-order Gaussian beam with linear polarization in the y direction.

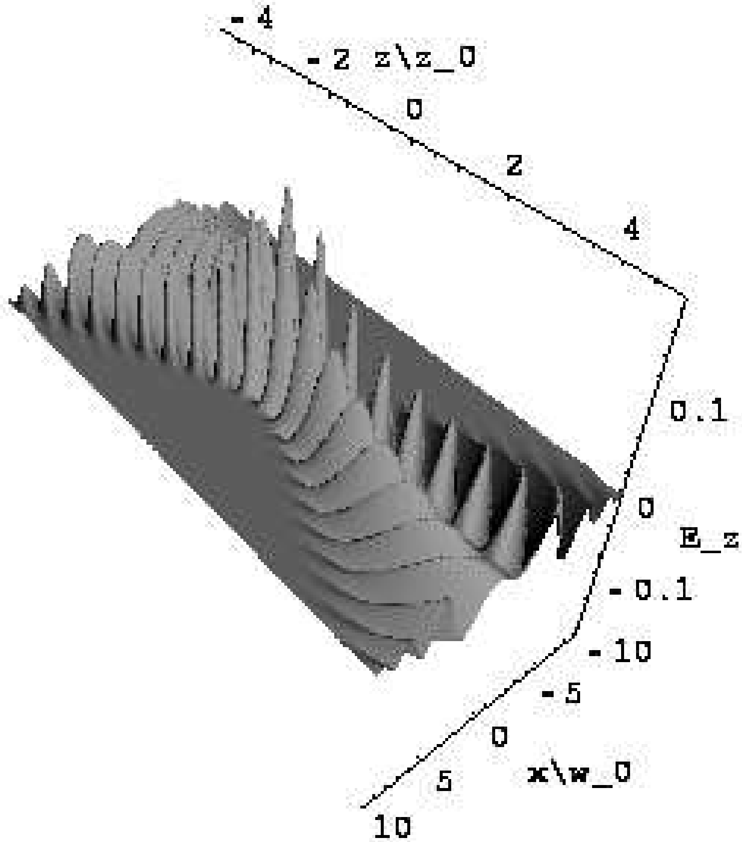

Figure 4: The electric field $E_z(x,0,z)$ of a linearly polarized Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (34).

2.5 The Lowest-Order Radially Polarized Beam

An advantage of our solution based on the vector potential is that we also can consider the case that only A_z is nonzero and has the form (25),

$$A_x = A_y = 0, \qquad A_z = \frac{E_0}{k\theta_0} f e^{-f\rho^2} g e^{i(kz-\omega t)}. \tag{37}$$

Then, the magnetic field is simply expressed in cylindrical coordinates as,

$$B_\perp = 0, \qquad B_\phi = E_0 \rho f^2 e^{-f\rho^2} g e^{i\varphi} + \mathcal{O}(\theta_0^2), \qquad B_z = 0, \tag{38}$$

and we find the electric field from eq. (30) to be,

$$\begin{aligned} E_\perp &= B_\phi = E_0 \rho f^2 e^{-f\rho^2} g e^{i\varphi} + \mathcal{O}(\theta_0^2), \\ E_\phi &= 0, \\ E_z &= i\theta_0 E_0 f^2 (1 - f\rho^2) e^{-f\rho^2} g e^{i\varphi} + \mathcal{O}(\theta_0^3). \end{aligned} \tag{39}$$

The fields E_x and E_z are shown in Figs. 5 and 6. The dislocation seen in Fig. 6 for $\rho \approx \varsigma$ is due to the factor $1 - f\rho^2$ that arises in the paraxial approximation, and would, I believe, be smoothed out on keeping higher-order terms in the expansion (16).

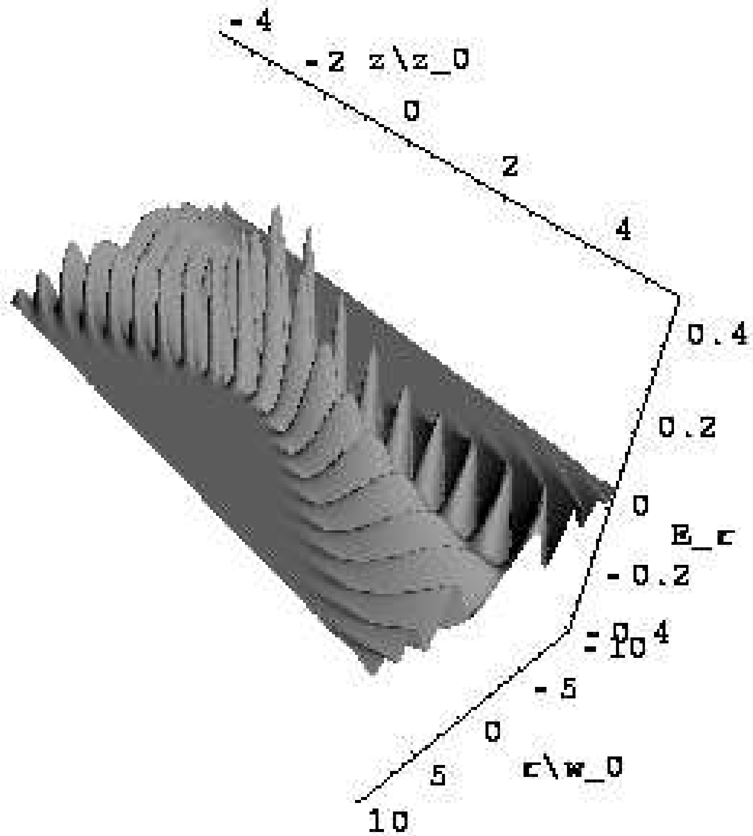

Figure 5: The electric field $E_r(r_\perp, 0, z)$ of a radially polarized Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (39).

The transverse electric field is radially polarized and vanishes on the axis. The longitudinal electric field is nonzero on the axis. Near the focus, $E_z \approx i\theta_0 E_0$ and the peak radial field is $E_0/\sqrt{2e} = 0.42E_0$. For large z, $E_\perp$ peaks at $\rho = \varsigma/\sqrt{2}$, corresponding to polar angle $\theta = \theta_0/\sqrt{2}$. For angles near this, $|E_\perp| \approx \rho\,|f|^2 \approx 1/z$, as expected in the far zone. In this region, the ratio of the longitudinal to transverse fields is $E_z/E_\perp \approx -i\theta_0 f\rho \approx -r_\perp/z$, as expected for a spherical wave front.

The factor f^2 in the fields implies a Gouy phase shift of $e^{-2i\tan^{-1}\varsigma}$, which is twice that of the lowest-order linearly polarized beams.

It is noteworthy that the simplest radially polarized mode (39)-(38) is not a member of the set of Gaussian modes based on Laguerre polynomials in cylindrical coordinates as reported in sec. 3.3b of [1]. This mode has, however, been discussed in [20].

Equations (38)-(39) describe a TM (transverse magnetic) mode. As is well known, corresponding to each TM wave solution to Maxwell's equations in free space, there is a TE (transverse electric) mode obtained by the duality transformation $\mathbf{E}_{\rm TE} = \mathbf{B}_{\rm TM}$, $\mathbf{B}_{\rm TE} = -\mathbf{E}_{\rm TM}$.

The TE waves can also be deduced from a vector potential, which we find to be,

$$A_\perp = 0, \qquad A_\phi = -\frac{E_0}{ik}\rho f^2 e^{-f\rho^2} g e^{i(kz-\omega t)}, \qquad A_z = 0, \tag{40}$$

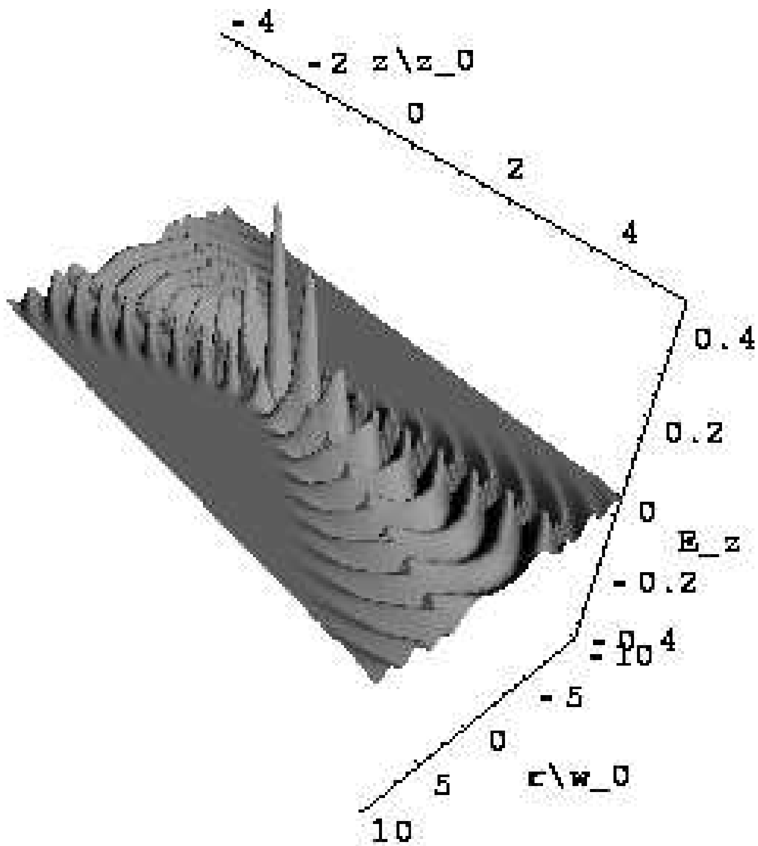

Figure 6: The electric field $E_z(r_\perp, 0, z)$ of an radially polarized Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (39).

after applying the duality transformation to eqs. (38)-(39). This vector potential does not consist of a single Cartestian component that satisfies a scalar wave equation. It is, however, a linear combination of Cartesian vector potentials A_x and A_y, each obeying the wave equation, that have second-order functional dependence on $r_\perp$ and z. Hence, this result provides a first glimpse of higher-order Gaussian modes, as are discussed elsewhere [20].

2.6 The Invariants $\mathbf{E} \cdot \mathbf{B}$ and $\mathbf{E}^2 - \mathbf{B}^2$

It is well known that there are only two distinct relativistic invariants that can be formed from the electric and magnetic fields, namely $\mathbf{E} \cdot \mathbf{B}$ and $\mathbf{E}^2 - \mathbf{B}^2$.

Both of these invariants vanish for a plane electromagnetic wave in vacuum.

Neither of these invariants vanish for the linearly polarized Gaussian beams of eqs. (32)-(33) because of the longitudinal field components E_z and B_z. However, the invariants do vanish along the z axis.

In the case of the radially polarized Gaussian beam of eqs. (38)-(39) the invariant $\mathbf{E} \cdot \mathbf{B}$ does vanish while the invariant $\mathbf{E}^2 - \mathbf{B}^2$ does not. In particular, $\mathbf{E}^2 - \mathbf{B}^2$ is nonzero along the z axis, and has the approximate value,

$$\mathbf{E}^2 - \mathbf{B}^2 \approx \theta_0^2 E_0^2 \sin^2(kz - \omega t) \tag{41}$$

for $z \ll z_0$, *i.e.*, near the focus.

A nonclassical interest in this result is that an electromagnetic field with positive $\mathbf{E}^2 - \mathbf{B}^2$ can "spark the vacuum" by spontaneous creation of electron-positron pairs if the value of that invariant approaches or exceeds the square of the so-called QED critical field strength,

$$E_{\rm crit} = \frac{m^2 c^3}{e\hbar} \approx 1.3 \times 10^{16}\ V/\text{cm}. \tag{42}$$

References

[1] H. Kogelnik and T. Li, *Laser Beams and Resonators*, Appl. Opt. **5**, 1550 (1966), http://kirkmcd.princeton.edu/examples/optics/kogelnik_ao_5_1550_66.pdf

[2] A.E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), chaps. 16-17, http://kirkmcd.princeton.edu/examples/EM/siegman_86.pdf

[3] P.W. Milonni and J.H. Eberly, *Lasers* (Wiley, 1988), sec. 14.5.

[4] A. Yariv, *Quantum Electronics*, 3rd ed. (Wiley, 1989), chap. 6, http://kirkmcd.princeton.edu/examples/QM/yariv_89.pdf

[5] K.T. McDonald, *Time Reversed Diffraction* (Sept. 5, 1999), http://kirkmcd.princeton.edu/examples/laserfocus.pdf

[6] J.A. Edighoffer and R.H. Pantell, *Energy exchange between free electrons and light in vacuum*, J. Appl. Phys. **50**, 6120 (1979), http://kirkmcd.princeton.edu/examples/accel/edighoffer_jap_50_6120_79.pdf

[7] F. Caspers and E. Jensen, *Particle Acceleration with the Axial Electric Field of a TEM_{10} Mode Laser Beam*, in *Laser Interaction and Related Plasma Phenomena*, ed. by H. Hora and G.H. Miley (Plenum Press, 1991), Vol. 9, pp. 459-466.

[8] E.J. Bochove, G.T. Moore and M.O. Scully, *Acceleration of particles by an asymmetric Hermite-Gaussian laser beam*, Phys. Rev. A **46**, 6640 (1992), http://kirkmcd.princeton.edu/examples/accel/bochove_pra_46_6640_92.pdf

[9] L.C. Steinhauer and W.D. Kimura, *A new approach to laser particle acceleration in vacuum*, J. Appl. Phys. **72**, 3237 (1992), http://kirkmcd.princeton.edu/examples/accel/steinhauer_jap_72_3237_92.pdf

[10] M.S. Zolotorev, S. Chattopadhyay and K.T. McDonald, *A Maxwellian Perspective on Particle Acceleration* (Feb. 24, 1998), http://kirkmcd.princeton.edu/examples/vacuumaccel.pdf

[11] G. Goubau and F. Schwering, *On the Guided Propagation of Electromagnetic Wave Beams*, IRE Trans. Ant. Prop. **AP-9**, 248 (1961), http://kirkmcd.princeton.edu/examples/optics/goubau_iretap_9_248_61.pdf

[12] G.D. Boyd and J.P. Gordon, *Confocal Multimode Resonator for Millimeter Through Optical Wavelength Masers*, Bell Sys. Tech. J. **40**, 489 (1961), http://kirkmcd.princeton.edu/examples/optics/boyd_bstj_40_489_61.pdf

[13] L.W. Davis, *Theory of electromagnetic beams*, Phys. Rev. A **19**, 1177 (1979), http://kirkmcd.princeton.edu/examples/optics/davis_pra_19_1177_79.pdf

[14] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part I (McGraw-Hill, 1953), pp. 115-116, http://kirkmcd.princeton.edu/examples/EM/morse_feshbach_v1.pdf

[15] J.B. Keller and W. Streifer, *Complex Rays with an Application to Gaussian Beams*, J. Opt. Soc. Am. **61**, 40 (1971), http://kirkmcd.princeton.edu/examples/EM/keller_josa_61_40_71.pdf

[16] G.A. Deschamps, *Gaussian Beam as a Bundle of Complex Rays*, El. Lett. **7**, 684 (1971), http://kirkmcd.princeton.edu/examples/EM/deschamps_el_7_684_71.pdf

[17] J.P. Barton and D.R. Alexander, *Fifth-order corrected electromagnetic field components for a fundamental Gaussian beam*, J. Appl. Phys. **66**, 2800 (1989), http://kirkmcd.princeton.edu/examples/optics/barton_jap_66_2800_89.pdf

[18] A. Sommerfeld, *Electrodynamics* (Academic Press, New York, 1952), secs. 22-23.

[19] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941), secs. 9.16-17, http://kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf

[20] L.W. Davis and G. Patsakos, *TM and TE electromagnetic beams in free space*, Opt. Lett. **6**, 22 (1981), http://kirkmcd.princeton.edu/examples/optics/davis_ol_6_22_81.pdf