Čerenkov Radiation in a Dielectric Wave Guide

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Consider a 2-dimensional simplification of the 3-dimensional phenomenon of Čerenkov radiation of a charge particle that propagates down a waveguide that is filled with a linear, isotropic dielectric of constant ϵ.

Two infinite, perfectly conducting parallel plates are separated by distance a. The gap between them is filled with a gas of index of refraction $n = \sqrt{\epsilon}$ (which may be taken as independent of frequency).

a) Give an expression for the guide wavelength λ_g and the phase velocity v_p of the possible guided waves in terms of the mode index m and the wavelength $\lambda = c/n\nu$ in an unbounded dielectric.

Suppose an infinite charged wire moves with velocity $v > c/n$ perpendicular to its length, and parallel to the conducting planes midway in the gap between them.

b) What waveguide modes are excited by the moving charged wire? Give a qualitative description of the frequency spectrum of the resulting waves.

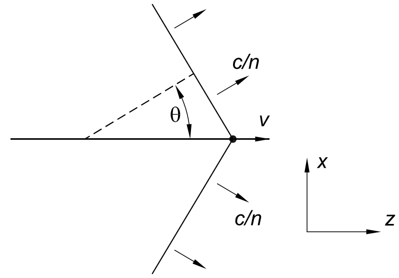
2 Solution

a) Waves cannot propagate between the plates with both the electric and magnetic fields transverse to the (average) axis of propagation, since that region is, in effect, inside a surrounding conducting surface. However, it is possible to have waves with either the electric field (TE modes) or the magnetic field (TM) modes transverse to the direction of propagation. To calculate the guide wavelength and the phase velocity, it suffices to consider either TE or TM modes.

Looking ahead to part b), we see that a line charge that moves down the guide perpendicular to its length with velocity $v = c/n$ emits Čerenkov radiation with a wedge angle given by,

$$\cos\theta = \frac{c}{nv}\,, \tag{1}$$

as shown below.



The electric field lines that emerge from the line charge are all compressed into the two sheets of the wedge, and point at angle θ to the x axis. The magnetic field, and not the electric field, is transverse is this case.

We take the z axis as the direction of propagation, and the x axis as perpendicular to the plates and $x = 0$ on the lower plate. We desire waves that have no dependence on y. Then, the general form of a TM wave is,

$$H_x = 0, \qquad H_y = H_0 \cos\frac{m\pi x}{a} e^{i(k_g z - \omega t)}, \qquad H_z = 0, \tag{2}$$

$$D_x = \frac{ck_g}{\omega} H_0 \cos\frac{m\pi x}{a} e^{i(k_g z - \omega t)}, \qquad D_y = 0, \qquad D_z = \frac{icm\pi}{a\omega} H_0 \sin\frac{m\pi x}{a} e^{i(k_g z - \omega t)}, \tag{3}$$

where m is the mode index, and the electric displacement field $\mathbf{D}$ has been obtained from the magnetic field $\mathbf{H}$ via the Maxwell equation,

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{4}$$

which holds in the medium between the parallel plates. The magnetic field cannot vary as $\sin m\pi x/a$ because then $D_z = \epsilon E_z$ would vary as $\cos m\pi x/a$, which does not vanish at $x = 0$ and a as it must. It is noteworthy that the fields between the plates are not simply the limit of those in a rectangular waveguide of width b in y as $b \to \infty$. This is because TM modes in a rectangular guide cannot have $D_y = 0$.

The magnetic field must also satisfy the wave equation in the dielectric medium, where the wave velocity is c/n,

$$\nabla^2 \mathbf{H} = \frac{n^2}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{5}$$

Plugging eq. (2) into (5), we obtain the dispersion relation,

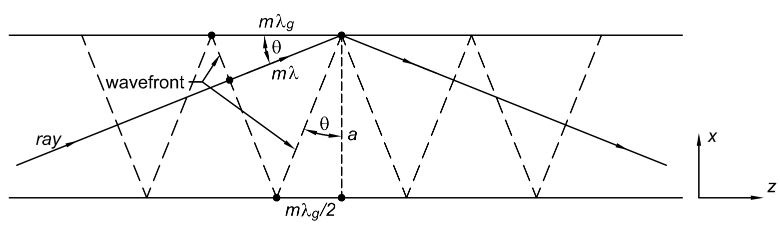
$$\left(\frac{m\pi}{a}\right)^2 + k_g^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{2\pi}{\lambda_g}\right)^2 = \left(\frac{n\omega}{c}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2, \tag{6}$$

where λ is the wavelength inside the dielectric medium. Thus,

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (m\lambda/2a)^2}}. \tag{7}$$

Waves can propagate only if $\lambda < 2a/m$.

It is helpful to have a geometric interpretation of these algebraic relations. While the guided waves are not transverse to the z axis, they can be thought of as due to a pattern of waves that are transverse to a ray that zig-zags down the guide, as shown in the figure.



For the mode of index m, the distance along the ray between successive wavefronts is $m\lambda$, and the distance along the z axis between successive wavefronts is $m\lambda_g$ (and not λ_g). To relate these two lengths geometrically, we note from the figure that the right triangle whose hypotenuse is $m\lambda_g$ is similar to the right triangle whose altitude is the gap height a. The base of the latter triangle is $m\lambda_g/2$, being half the period of the zig-zag wave. Hence,

$$\frac{m\lambda_g}{m\lambda} = \frac{\sqrt{a^2 + (m\lambda_g/2)^2}}{a}\,, \tag{8}$$

and again,

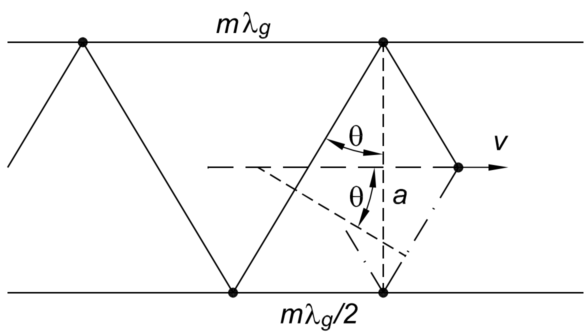
$$\lambda_g = \frac{\lambda}{\sqrt{1 - (m\lambda/2a)^2}}\,. \tag{9}$$

The phase velocity is therefore,

$$v_p = \nu\lambda_g = \frac{c}{n}\frac{\lambda_g}{\lambda} = \frac{c}{n}\sqrt{1 + \left(\frac{m\lambda_g}{2a}\right)^2} = \frac{c}{n}\frac{1}{\sqrt{1 - (m\lambda/2a)^2}}\,. \tag{10}$$

For λ and λ_g small enough, the phase velocity can be less than c, and the longitudinal electric field, $E_z = D_z/\epsilon$, of the guided waves can accelerate charges that may be moving down the guide with velocity $v = v_p$. See, W.D. Kimura *et al.*, Phys. Rev. Lett. **74**, 546 (1995), http://kirkmcd.princeton.edu/examples/optics/kimura_prl_74_546_95.pdf

b) The Čerenkov radiation reflects off the conducting plates and sets up a criss-cross pattern like that of a guided wave mode:



Recalling the figure on p. 2, we see that if this pattern corresponds to a mode of index m, then the distance between successive wavefronts (of, say, the upper branch of the Čerenkov wave) along the guide is $m\lambda_g$. The upper angle in the triangle with altitude a is the Čerenkov angle θ. Hence,

$$\frac{m\lambda_g}{2a} = \tan\theta = \frac{\sqrt{1 - (c/nv)^2}}{c/nv} = \sqrt{\left(\frac{nv}{c}\right)^2 - 1}. \tag{11}$$

The phase velocity of this mode follows from eqs. (10) and, (11) as

$$v_p = \frac{c}{n}\sqrt{1 + \left(\frac{m\lambda_g}{2a}\right)^2} = v, \tag{12}$$

for any index m. Hence, the Čerenkov effect can excite all waveguide modes.

In principle, the Čerenkov wavefront is extremely narrow in time, so that its frequency spectrum is essentially flat – although a discrete spectrum of multiples of the fundamental,

$$\nu = \frac{c}{2an\sqrt{1 - (c/nv)^2}}\,. \tag{13}$$

In practice, the frequency spectrum extends up only to the characteristic resonant frequencies of the dielectric medium, typically in the near ultraviolet, beyond which the dielectric "constant" drops toward unity (or less than unity above the plasma frequency) and the Čerenkov effect disappears. The thickness of the sheet of electric field in the Čerenkov wedge is approximately the shortest wavelength for which the index of refraction still has value n.

See sec. 5 of http://kirkmcd.princeton.edu/examples/weizsacker.pdf for a discussion of the so-called formation length for Čerenkov radiation.