The Helical Wiggler

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1 Problem

A variant on the electro- or magnetostatic boundary value problem arises in accelerator physics, where a specified field, say $\mathbf{B}(0,0,z)$, is desired along the z axis. In general, there exist static fields $\mathbf{B}(x,y,z)$ that reduce to the desired field on the axis, but the "boundary condition" $\mathbf{B}(0,0,z)$ is not sufficient to insure a unique solution.[1]

For example, find a field $\mathbf{B}(x,y,z)$ that reduces to,

$$\mathbf{B}(0,0,z) = B_0 \cos kz\ \hat{\mathbf{x}} + B_0 \sin kz\ \hat{\mathbf{y}} \tag{1}$$

on the z axis. In this, the magnetic field rotates around the z axis as z advances.

Show how the use of rectangular or cylindrical coordinates leads "naturally" to different forms for $\mathbf{B}$.

One 3-dimensional field extension of (1) is the so-called helical wiggler [2, 3], which obeys the auxiliary requirement that the field at $z+\delta$ be the same as the field at z, but rotated by angle $k\delta$. Show that this field pattern can be realized by a current-carrying wire that is wound in a helix of period $\lambda = 2\pi/k$ [4].

2 Solution

2.1 Solution in Rectangular Coordinates

We first seek a solution in rectangular coordinates, and expect that separation of variables will apply. Thus, we consider the form,

$$\begin{aligned} B_x &= f(x)g(y)\cos kz, && (2)\\ B_x &= F(x)G(y)\sin kz, && (3)\\ B_z &= A(x)B(y)C(z). && (4) \end{aligned}$$

Then,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 = f'g\cos kz + FG'\sin kz + ABC', \tag{5}$$

where the $'$ indicates differentiation of a function with respect to its argument. Equation (5) can be integrated with respect to z to give,

$$ABC = -\frac{f'g}{k}\sin kz + \frac{FG'}{k}\cos kz. \tag{6}$$

[1]If the axial field has only an axial component a unique solution obtains [1].

The z component of $\nabla \times \mathbf{B} = 0$ tells us that,

$$\frac{\partial B_x}{\partial y} = f g' \cos kz = \frac{\partial B_y}{\partial x} = F' G \sin kz. \tag{7}$$

For this to hold at all x and y we must have $g' = 0 = F'$, which implies that g and F are constant, say 1. Likewise,

$$\frac{\partial B_x}{\partial z} = -fk \sin kz = \frac{\partial B_z}{\partial x} = A'BC = -\frac{f''}{k} \sin kz, \tag{8}$$

using eqs. (6)-(7). Thus, $f'' - k^2 f = 0$, so,

$$f = f_1 e^{kx} + f_2 e^{-kx}. \tag{9}$$

Finally,

$$\frac{\partial B_y}{\partial z} = Gk \cos kz = \frac{\partial B_z}{\partial y} = AB'C = \frac{G''}{k} \sin kz, \tag{10}$$

so,

$$G = G_1 e^{ky} + G_2 e^{-ky}. \tag{11}$$

The "boundary conditions" $f(0) = B_0 = G(0)$ are satisfied by,

$$f = B_0 \cosh kx, \qquad G = B_0 \cosh ky, \tag{12}$$

which together with eq. (6) leads to the solution,

$$\begin{aligned} B_x &= B_0 \cosh kx \cos kz, && (13) \\ B_y &= B_0 \cosh ky \sin kz, && (14) \\ B_z &= -B_0 \sinh kx \sin kz + B_0 \sinh ky \cos kz, && (15) \end{aligned}$$

This satisfies the last "boundary condition" that $B_z(0, 0, z) = 0$.

However, this solution does not have helical symmetry.

2.2 Solution in Cylindrical Coordinates

Suppose instead, we look for a solution in cylindrical coordinates (r, θ, z). We again expect separation of variables, but we seek to enforce the helical symmetry that the field at $z + \delta$ be the same as the field at z, but rotated by angle $k\delta$. This symmetry implies that the argument kz should be replaced by $kz - \theta$, and that the field has no other θ dependence.

We begin constructing our solution with the hypothesis that,

$$\begin{aligned} B_r &= F(r) \cos(kz - \theta), && (16) \\ B_\theta &= G(r) \sin(kz - \theta). && (17) \end{aligned}$$

To satisfy the condition (1) on the z axis, we first transform this to rectangular components,

$$\begin{aligned} B_z &= F(r) \cos(kz - \theta) \cos\theta + G(r) \sin(kz - \theta) \sin\theta, && (18) \\ B_y &= -F(r) \cos(kz - \theta) \sin\theta + G(r) \sin(kz - \theta) \cos\theta, && (19) \end{aligned}$$

from which we learn that the "boundary conditions" on F and G are.

$$F(0) = G(0) = B_0. \tag{20}$$

A suitable form for B_z can be obtained from $(\boldsymbol{\nabla} \times \mathbf{B})_r = 0$:

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} = \frac{\partial B_\theta}{\partial z} = kG\cos(kz - \theta), \tag{21}$$

so,

$$B_z = -krG\sin(kz - \theta), \tag{22}$$

which vanishes on the z axis as desired.

From either $(\boldsymbol{\nabla} \times \mathbf{B})_\theta = 0$ or $(\boldsymbol{\nabla} \times \mathbf{B})_z = 0$ we find that,

$$F = \frac{d(rG)}{dr} = \frac{d(krG)}{dkr}. \tag{23}$$

Then, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ leads to,

$$(kr)^2\frac{d^2(krG)}{d(kr)^2} + kr\frac{d(krG)}{d(kr)} - [1 + (kr)^2](krG) = 0. \tag{24}$$

This is the differential equation for the modified Bessel function of order 1 [5]. Hence,

$$G = C\frac{I_1(kr)}{kr} = \frac{C}{2}\left[1 + \frac{(kr)^2}{8} + \cdots\right], \tag{25}$$

$$F = C\frac{dI_1}{d(kr)} = C\left(I_0 - \frac{I_1}{kr}\right) = \frac{C}{2}\left[1 + \frac{3(kr)^2}{8} + \cdots\right]. \tag{26}$$

The "boundary conditions" (20) require that $C = 2B_0$, so our second solution is,

$$B_r = 2B_0\left(I_0(kr) - \frac{I_1(kr)}{kr}\right)\cos(kz - \theta), \tag{27}$$

$$B_\theta = 2B_0\frac{I_1}{kr}\sin(kz - \theta), \tag{28}$$

$$B_z = -2B_0 I_1 \sin(kz - \theta), \tag{29}$$

which is the form discussed in [3].

2.3 Magnetic Field Due to a Double Helix

This section follows [6].

We consider a wire that carries current I and is wound in the form of a helix of radius a and period $\lambda = 2\pi/k$. A suitable equation of this helix is,

$$x_1 = a\sin kz, \qquad y_1 = -a\cos kz. \tag{30}$$

The magnetic field due to this winding has a nonzero z component along the axis, which is not desired. Therefore, we also consider a second helical winding,

$$x_2 = -a \sin kz, \qquad y_2 = a \cos kz, \tag{31}$$

which is offset from the first by half a period and which carries current $-I$. The combined magnetic field from the two helices has no component along their common axis.

The unit vector $\hat{\mathbf{l}}_{1,2}$ that is tangent to helix 1(2) at a point,

$$\mathbf{r}'_{1,2} = (x'_{1,2}, y'_{1,2}, z') = (\pm a \sin kz', \mp a \cos kz', z') \tag{32}$$

has components,

$$\hat{\mathbf{l}}_{1,2} = \frac{(\pm 2\pi a \cos kz', \pm 2\pi a \sin kz', \lambda)}{\sqrt{\lambda^2 + (2\pi a)^2}}, \tag{33}$$

and the element $d\mathbf{l}'_{1,2}$ of arc length along the helix is related by,

$$d\mathbf{l}'_{1,2} = \hat{\mathbf{l}}'_{1,2} dz' \frac{\sqrt{\lambda^2 + (2\pi a)^2}}{\lambda} = dz'(\pm ka \cos kz', \pm ka \sin kz', 1). \tag{34}$$

The magnetic field $\mathbf{B}$ at a point $\mathbf{r} = (0, 0, z)$ on the axis is given by,

$$\begin{aligned}
\mathbf{B}(0,0,z) &= \frac{I}{c}\int_1 \frac{d\mathbf{l}'_1 \times (\mathbf{r}'_1 - \mathbf{r})}{|\mathbf{r}'_1 - \mathbf{r}|^3} - \frac{I}{c}\int_2 \frac{d\mathbf{l}'_2 \times (\mathbf{r}'_2 - \mathbf{r})}{|\mathbf{r}'_2 - \mathbf{r}|^3} \\
&= \frac{2Ia}{c}\int_{-\infty}^{\infty} \frac{dz'}{[a^2 + (z'-z)^2]^{3/2}} [\hat{\mathbf{x}}(k(z'-z)\sin kz' + \cos kz') \\
&\qquad\qquad + \hat{\mathbf{y}}(-k(z'-z)\cos kz' + \sin kz')] \\
&= \frac{2I}{ca}\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/2}} [\hat{\mathbf{x}}(kat \sin(kat + kz) + \cos(kat + kz)) \\
&\qquad\qquad + \hat{\mathbf{y}}(-kat \cos(kat + kz) + \sin(kat + kz))] \\
&= \frac{4Ik}{c}(\hat{\mathbf{x}} \cos kz + \hat{\mathbf{y}} \sin kz)\left[\frac{1}{ka}\int_0^{\infty} \frac{\cos kat}{(1+t^2)^{3/2}} dt + \int_0^{\infty} \frac{t \sin kat}{(1+t^2)^{3/2}} dt\right],
\end{aligned} \tag{35}$$

where we made the substitution $z' - z = at$ in going from the second line to the third. Equation 9.6.25 of [5] tells us that,

$$\int_0^{\infty} \frac{\cos kat}{(1+t^2)^{3/2}} dt = ka K_1(ka), \tag{36}$$

where K_1 also satisfies eq. (24). We integrate the last integral by parts, using,

$$u = \sin kat, \qquad dv = \frac{t\ dt}{(1+t^2)^{3/2}}, \qquad \text{so} \qquad du = ka \cos kat\ dt, \qquad v = -\frac{1}{\sqrt{1+t^2}}. \tag{37}$$

Thus,

$$\int_0^{\infty} \frac{t \sin kat}{(1+t^2)^{3/2}} dt = ka \int_0^{\infty} \frac{\cos kat}{\sqrt{1+t^2}} dt = ka K_0(ka), \tag{38}$$

using 9.6.21 of [5]. Hence,

$$\mathbf{B}(0,0,z) = \frac{4Ik}{c}\,[kaK_0(ka) + K_1(ka)]\,(\hat{\mathbf{x}}\cos kz + \hat{\mathbf{y}}\sin kz). \tag{39}$$

Both $K_0(ka)$ and $K_1(ka)$ have magnitudes $\approx 0.5e^{-ka}$ for $ka \approx 1$. That is, the field on the axis of the double helix is exponentially damped in the radius a for a fixed current I.

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