Neutral-Pion Decay

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(September 15, 1976; updated June 4, 2019)

1 Problem

Consider the decay of the neutral π meson of (total) energy E_π to two photons, $\pi^0 \to \gamma\gamma$.

1. If the two photons are observed in the laboratory with energies E_1 and E_2 and angle α between them, what is their invariant mass?
2. If the decay of the π^0 is isotropic in its rest frame, what is the laboratory distribution dN/dE_γ of the energies of the decay photons?
3. What is the minimum opening angle, $\alpha_{\rm min}$, between the two photons in the lab frame?
4. What is the distribution $dN/d\alpha$ of the opening angle between the two photons in the lab frame?
5. If the two photons are detected at positions x_1 and x_2 in a plane perpendicular to the direction of the π^0 at a distance D, what is the projected impact point x of the π^0 had it not decayed? You may assume that $|x_1 - x_2| \ll D$, which is true for most, but not quite all, decays if $E_\pi/m_\pi \gg 1$.
6. What is the maximum laboratory angle $\theta_{\rm max}$ between the direction of a photon from π^0 decay and the direction of the π^0, supposing the photon is observed to have energy $E_\gamma \gg m_\pi$?
7. Suppose π^0's are produced in some scattering process with distribution $N_\pi(E_\pi, \theta_\pi)$, where angle θ_π is measured with respect to the beam direction. That is, $N_\pi(E_\pi, \theta_\pi)\, dE_\pi\, d\Omega_\pi$ is the number of π^0's in energy interval dE_π centered about energy E_π that point towards solid angle $d\Omega_\pi$ centered about angles (θ_π, ϕ_π). A detector is placed at angle θ to the beam and records the energy spectrum $N_\gamma(E_\gamma, \theta)$ of the photons that strike it. Show that the π^0 spectrum can be related to the photon spectrum by,
$$N_\pi(E_\pi, \theta) = -\frac{E_\pi}{2} \frac{dN_\gamma(E_\gamma = E_\pi, \theta)}{dE_\gamma}\,, \tag{1}$$
if $E_\pi \gg m_\pi$.

2 Solution

1. Since a (real) photon has no mass, its energy and momentum are the same: $E_\gamma = P_\gamma$.

In this part we suppose that photon 1 propagates along the $+z$ axis, so its energy-momentum 4-vector can be written (in units where $c = 1$) as,

$$q_1 = (E, P_x, P_y, P_z) = (E_1, 0, 0, E_1). \tag{2}$$

We can define photon 2 to be moving in the x-z plane, so its 4-vector is ,

$$q_2 = (E_2, E_1 \sin\alpha, , 0, E_1 \cos\alpha). \tag{3}$$

The invariant mass of the two photons is related by,

$$\begin{aligned} m^2 &= (q_1 + q_2)^2 = q_1^2 + q_2^2 + 2q_1 \cdot q_2 = 0 + 0 + 2E_1E_2(1 - \cos\alpha) \\ &= 4E_1E_2 \sin^2\alpha/2. \end{aligned} \tag{4}$$

If we had defined the π^0 to propagate along the $+z$ axis, we could still define the decay plane to be the x-z plane and write,

$$q_1 = (E_1, E_1 \sin\theta_1, 0, E_1 \cos\theta_1), \qquad q_2 = (E_2, -E_2 \sin\theta_2, 0, E_1 \cos\theta_2), \tag{5}$$

so that,

$$m^2 = (q_1 + q_2)^2 = 2E_1E_2(1 - \cos(\theta_1 + \theta_2)) = 4E_1E_2 \sin^2\alpha/2, \tag{6}$$

where the opening angle is $\alpha = \theta_1 + \theta_2$.

2. In this part we suppose the π^0 propagates along the $+z$ axis, and we define $\theta^\star$ as the angle of photon 1 to the z axis in the rest frame of the π^0.

The decay is isotropic in the rest frame, so the distribution is flat as a function of $\cos\theta^\star$. We write,

$$\frac{dN}{d\cos\theta^\star} = \frac{1}{2}, \tag{7}$$

normalized to unity over the interval $-1 \le \cos\theta^\star \le 1$. The desired distribution of photon energies can be related to this via,

$$\frac{dN}{dE_\gamma} = \frac{dN}{d\cos\theta^\star}\frac{d\cos\theta^\star}{dE_\gamma} = \frac{1}{2}\frac{d\cos\theta^\star}{dE_\gamma}. \tag{8}$$

To relate E_γ to $\cos\theta^\star$, we examine the transformation between the lab frame and the rest frame of the π^0, for which the boost is described by the Lorentz factors $\gamma = E_\pi/m_\pi$ and $\beta = v_\pi/c = P_\pi c/E_\pi$ (although we use units where $c = 1$).

This procedure is useful for any two-body decay, $a \to b + c$, of a spin-0 particle a.

We consider the rest frame of particle a, for which the boost is described by the Lorentz factors $\gamma = E_a/m_a$ and $\beta = v_a/c = P_a c/E_a$ (although we use units where $c = 1$).

Energy-momentum conservation in the decay can be expressed as,

$$E_{a,\mu} = E_{b,\mu} + E_{c,\mu}. \tag{9}$$

To emphasize the details of particle b, we isolate the energy-momentum 4-vector of particle c, whose square is m_c^2,

$$E_{c,\mu} = E_{a,\mu} - E_{b,\mu}, \tag{10}$$
$$m_c^2 = E_{c,\mu}^2 = (E_{a,\mu} - E_{b,\mu})^2 = m_a^2 + m_b^2 - 2E_{a,\mu}E_b^\mu \tag{11}$$

In the rest frame of particle a, labeled by superscript *, $E_{a,\mu}^\star = (m_a, \mathbf{0})$ while $E_{b,\mu}^\star = (E_b^\star, \mathbf{P}_b^\star)$, so $E_{a,\mu}E_b^\mu = m_a E_b^\star$, and,

$$E_b^\star = \frac{m_a^2 + m_b^2 - m_c^2}{2m_a}, \qquad P_b^\star = \sqrt{E_b^\star - m_b^2} \ (= P_c^\star) \,. \tag{12}$$

The lab-frame energy of particle b is given by,

$$E_b = \gamma E_b^\star + \gamma\beta P_{b,z}^\star = \gamma(E_b^\star + \beta P_b^\star \cos\theta^\star). \tag{13}$$

Then,

$$\frac{dE_b}{d\cos\theta^\star} = \gamma\beta P_b^\star = \frac{P_a P_b^\star}{m_a}\,, \tag{14}$$

and the energy distribution follows from eq. (8) as,

$$\frac{dN}{dE_b} = \frac{dN}{d\cos\theta^\star}\frac{d\cos\theta^\star}{dE_b} = \frac{1}{2}\frac{d\cos\theta^\star}{dE_b}. = \frac{1}{2\gamma\beta P_b^\star} = \frac{m_a}{2P_a P_b^\star}\,. \tag{15}$$

The distribution is flat, with limiting values of $\gamma(E_b^\star \pm P_b^\star)/2$, according to eq. (13).

For the decay $\pi^0 \to \gamma\gamma$, we have $m_a = m_{\pi^0}$, $m_b = m_c = 0$, $E_\gamma^\star = P_\gamma^\star = m_{\pi^0}/2$, and since particles b and c are both photons,

$$\frac{dN}{dE_\gamma} = 2\frac{dN}{dE_b} = \frac{m_{\pi^0}}{P_{\pi^0}P_\gamma^\star} = \frac{2}{P_{\pi^0}}\,, \tag{16}$$

with decay photons of energies $0 \le E_\gamma \le E_{\pi^0}$.[1]

3. Since the two decay products have equal mass (zero), the minimum decay angle in the lab occurs at either $\cos\theta^\star = 1$ or 0. If $\cos\theta^\star = 1$, one of the photons goes forward, and the other goes backwards. Since the mass of the photon is zero, its backwards velocity is c, and the boost of the pion to the lab frame cannot overcome this. The opening angle between the two photons is then π, a maximum rather than a minimum. (If the decay products have mass, it is possible that the velocity of the backward going particles is less than that of the parent, and both particles go forward in the lab, with minimum opening angle of zero.)

   We conclude that the minimum opening angle $\alpha_{\min}$ occurs for the symmetric decay, $\cos\theta^\star = 0$. In this case, the transverse momentum of the photons is $m_\pi/2$, both in lab

[1] For the decay $\pi^+ \to \mu^+\nu_\mu$ with $m_{\pi^+} = 139.6$ MeV/c^2, $m_\mu = 105.7$ MeV/c^2 and $m_\nu \approx 0$, we have $E_\nu^\star$ ($\approx P_\nu^\star = P_\mu^\star$) $= 29.8$ MeV $\approx 0.21 m_{\pi^+}$, $E_\mu^\star = 109.8$ MeV $\approx 0.79 m_{\pi^+}$, so the laboratory distributions of the neutrino and muons energies are flat with $0 < E_\nu < 0.42E_\pi$ and $0.58E_\pi < E_\mu < E_\pi$.

frame and the π^0 rest frame. In the lab frame, a photon" total momentum equals its total energy, which is just $E_\pi/2$ for the symmetric decay. Hence,

$$\sin\frac{\alpha_{\rm min}}{2} = \frac{m_\pi}{E_\pi} = \frac{1}{\gamma}\,. \tag{17}$$

4. The distribution of decays in opening angle α can be written as,

$$\frac{dN}{d\alpha} = \frac{dN}{d\cos\theta^\star}\frac{d\cos\theta^\star}{d\alpha} = \frac{1}{2}\frac{d\cos\theta^\star}{d\alpha}\,, \tag{18}$$

recalling eq. (7).

One way to relate $\alpha = \theta_1 + \theta_2$ and $\cos\theta^\star$ is to recall eq. (4) and that from eq. (13) the two photon energies are $E_{1,2} = \gamma(m_\pi/2)(1 \pm \beta\cos\theta^\star)$,

$$\sin^2\alpha/2 = \frac{m_\pi^2}{4E_1E_2} = \frac{1}{\gamma^2(1-\beta^2\cos^2\theta^\star)}\,, \tag{19}$$

or,

$$\cos\theta^\star = \frac{1}{\beta}\sqrt{1-\frac{1}{\gamma^2\sin^2\alpha/2}} = \frac{\sqrt{\gamma^2\sin^2\alpha/2-1}}{\gamma\beta\sin\alpha/2}\,. \tag{20}$$

Taking the derivative, we use eq. (18) to find,

$$\frac{dN}{d\alpha} = \frac{1}{4\gamma\beta}\frac{\cos\alpha/2}{\sin^2\alpha/2}\frac{1}{\sqrt{\gamma^2\sin^2\alpha/2-1}}. \tag{21}$$

This distribution is peaked at $\alpha_{\rm min}$ where $\sin\alpha_{\rm min}/2 = 1/\gamma$, and vanishes at $\alpha_{\rm max} = \pi$.

A subtle issue is revealed on integration of eq. (21), letting $x = \gamma\sin\alpha/2$, so that,

$$\int_{\alpha_{\rm min}}^{\pi}\frac{dN}{d\alpha}\,d\alpha = \frac{1}{2\beta}\int_1^\gamma\frac{dx}{x^2\sqrt{x^2-1}} = \frac{1}{2\beta}\frac{\sqrt{\gamma^2-1}}{\gamma} = \frac{1}{2}\,, \tag{22}$$

using Dwight 282.01. The integral is only 1/2, rather than 1, because, as the decay angle $\theta^\star$ in the pion rest frame varies from 0 to π, the lab-frame opening angle varies from $\alpha_{\rm rm}$ at $\theta^\star = 0$ up to π (for $\theta^\star = \pi/2$) and then back down to $\alpha_{\rm min}$ at $\theta^\star = \pi$. That is, $\theta^\star$ is a double-valued function of α, so integration (once) over α includes only half of the total decays.

If it is desired that the distribution $dN/d\alpha$ be normalized to unity, eq. (21) should be multiplied by 2.

5. The transverse momenta of the two decay photons (with respect to the lab direction of the π^0) are equal and opposite. When the observed separation of the two photons obeys $|x_1 - x_2| \ll D$, the angles of the photons with respect to the direction of the π^0 are small, and the transverse momenta can be written as,

$$P_i\frac{x_i - x}{D} = E_i\frac{x_i - x}{D}\,, \tag{23}$$

Hence,

$$E_1(x_1 - x) = E_2(x - x_2), \tag{24}$$

and the virtual impact point of the π^0 is,

$$x = \frac{x_1 E_1 + x_2 E_2}{E_1 + E_2} = \frac{x_1 E_1 + x_2 E_2}{E_\pi} . \tag{25}$$

6. The transverse momentum of a decay photon with respect to the direction of the π^0 is,

$$P_\perp = P_\gamma \sin\theta = E_\gamma \sin\theta, \tag{26}$$

where θ is the angle between the direction of the photon and the π^0. This quantity is invariant with respect to the boost to the rest frame of the π^0, so,

$$P_\perp = P_\perp^\star = P_\gamma^\star \sin\theta^\star = \frac{m_\pi}{2} \sin\theta^\star. \tag{27}$$

Comparing eqs. (26) and (27) we see that,

$$\sin\theta = \frac{m_\pi}{2E_\gamma} \sin\theta^\star. \tag{28}$$

So long as $\theta \le \pi/2$, we find that,

$$\sin\theta_{\rm max} = \frac{m_\pi}{2E_\gamma} , \tag{29}$$

and for $E_\gamma \gg m_\pi$,

$$\theta_{\rm max} \approx \frac{m_\pi}{2E_\gamma} . \tag{30}$$

However, there are cases when $\theta > \pi/2$, for which $P_\parallel = P_\gamma \cos\theta < 0$. Recalling the boost formalism of item 2 above,

$$P_\parallel = \gamma_\pi (P_\parallel^\star + \beta_\pi E^\star) = \frac{\gamma_\pi m_\pi}{2} (\cos\theta^\star + \beta_\pi), \tag{31}$$

we see that $P_\parallel = 0$ and $\theta = \pi/2$ when $\cos\theta^\star = -\beta_\pi$. In this case,

$$E_\gamma = P_\perp = \frac{m_\pi}{2}\sqrt{1-\beta_\pi^2} = \frac{m_\pi}{2\gamma_\pi} = \frac{m_\pi^2}{2E_\pi} < \frac{m_\pi}{2}, \tag{32}$$

since $E_\pi \ge m_\pi$. Thus, the result (29) holds for $E_\gamma > m_\pi/2$.

7. We will use information about the photon spectrum for energies $E_\gamma \gg m_\pi$, so the maximum angle between the photon and its parent π^0 is negligibly small, according to the result of part 6. Then, the probability that a photon hits a detector of a fixed solid angle is the same as the probability that its parent π^0 would have hit the detector, had the π^0 not decayed. That is, we can ignore any possible complication due to solid angle transformation between the π^0 and the photon.

According to eq. (16), the number $N_\gamma(E_\gamma)$ of photons that appear in energy interval dE_γ due to the decay of a single π^0 is,

$$N_\gamma = \frac{2}{P_\pi} \approx \frac{2}{E_\pi}\,, \tag{33}$$

where the factor of 2 occurs because two photon are produced in each decay, and the approximation holds when $E_\pi \gg m_\pi$ so that it certainly applies when $E_\gamma \gg m_\pi$.

If π^0's are produced with an energy spectrum $N_\pi(E_\pi, \theta_\pi)$, then the energy spectrum of the decay photons observed in a detector centered on θ_π is related by,

$$N_\gamma(E_\gamma, \theta_\gamma = \theta_\pi) = \int_{E_\gamma}^\infty \frac{2}{E_\pi} N_\pi(E_\pi, \theta_\pi)\ dE_\pi. \tag{34}$$

Taking the derivative, we find,

$$N_\pi(E_\pi, \theta_\pi) = -\frac{E_\pi}{2} \frac{dN_\gamma(E_\gamma = E_\pi, \theta_\gamma = \theta_\pi)}{dE_\gamma}\,. \tag{35}$$

A more detailed discussion of this problem has been given by R.M. Sternheimer, *Energy Distribution of γ Rays from π^0 Decay*, Phys. Rev. **99**, 277 (1955),
http://kirkmcd.princeton.edu/examples/detectors/sternheimer_pr_99_277_55.pdf

For a discussion of the slightly more complicated case of $\pi^\pm$ decay, see
http://kirkmcd.princeton.edu/examples/offaxisbeam.pdf