An RF Cavity
in Which Transverse Fields Grow Linearly with Radius

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1 Problem

A simple rf cavity is a right circular cylinder of radius a and length d, for which the $\mathrm{TM}_{0,1,0}$ mode has electromagnetic fields (in Gaussian units),

$$E_z(r,\theta,z,t) = E_0 J_0(kr)\cos\omega t, \tag{1}$$

$$B_\theta(r,\theta,z,t) = E_0 J_1(kr)\sin\omega t, \tag{2}$$

where $ka = 2.405$ is the first zero of the Bessel function J_0.

Such a cavity is potentially interesting for particle acceleration in that the electric field points only along the axis and is independent of z, so that a large fraction of the maximal energy eEd could be imparted to a particle of charge e as it traverses the cavity. However, such cavities are not useful in practice for at least two reasons: the particles must pass through the cavity wall to enter or exit the cavity and thereby suffer undesirable scattering; the magnetic field does not vary linearly with radius, and so acts like a nonlinear lens for particles whose motion is not exactly parallel to the axis.

Practical accelerating cavities have apertures (irises) of radius b in the entrance and exit surfaces, so that a beam of particles can pass through without encountering any material. In this case, the electric field can no longer be purely axial. Deduce the simplest electromagnetic mode of a cavity with apertures for which the transverse components of the electric and magnetic fields vary linearly with radius. Deduce also the shape of the wall of a perfectly conducting cavity that could support this mode.

Consider a cavity of extent $-d < z < d$, with azimuthal symmetry and symmetry about the plane $z = 0$, that could be a unit cell of a repetitive structure. This implies that either $E_z = 0$ at $(r, z) = (0, d)$ and $(0, -d)$, or $\partial E_z/\partial z = 0$ at these points.

2 Solution

We seek a standing-wave solution where, say, the time dependence of E_z is $\cos\omega t$. The cavity is symmetric about the plane $z = 0$, so we expect the z dependence of E_z to have the form $\cos k_n z$, where,

$$k_n = \begin{cases} (2n-1)\pi/2d, & \text{if } E_z(0,-d) = E_z(0,d) = 0, \\ n\pi/d, & \text{if } \partial E_z(0,-d)/\partial z = \partial E_z(0,d)/\partial z = 0. \end{cases} \tag{3}$$

We can combine these two cases in the notation,

$$k_n = (2n - n_0)\frac{\pi}{2d}, \text{ where } \begin{cases} n_0 = 1, & \text{if } E_z(0, -d) = E_z(0, d) = 0, \\ n_0 = 2, & \text{if } \partial E_z(0, -d)/\partial z = \partial E_z(0, d)/\partial z = 0. \end{cases} \tag{4}$$

where $n = 1, 2, 3, ...$

Our trial solution,

$$E_z(r, z, t) = f_n(r) \cos k_n z \cos \omega t, \tag{5}$$

must satisfy the wave equation,

$$\nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial f_n}{r}\right) - \left(k_n^2 - \frac{\omega^2}{c^2}\right) f_n = 0. \tag{6}$$

This is the differential equation for the modified Bessel function of order zero, $I_0(K_n r)$, where,

$$K_n^2 = k_n^2 - \frac{\omega^2}{c^2} = \left[(2n - n_0)\frac{\pi}{2d}\right]^2 - \left(\frac{2\pi}{\lambda}\right)^2, \tag{7}$$

the free-space wavelength at frequency ω is $\lambda = 2\pi c/\omega$, and,

$$I_0(x) = 1 + (x/2)^2 + \frac{(x/2)^4}{(2!)^2} + \frac{(x/2)^6}{(3!)^2} + \cdots \tag{8}$$

In the special case of $k_n = 0$, (6) reverts to the equation for the ordinary Bessel function J_0, and the fields (1-2) are obtained. Since this form cannot exist in a cavity with apertures, we ignore it in further discussion.

A Fourier series for E_z with nonzero k_n is then,

$$E_z(r, z, t) = \sum_{n=1}^{\infty} a_n I_0(K_n r) \cos k_n z \cos \omega t. \tag{9}$$

The radial component of the electric field is obtained from,

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial r E_r}{\partial r} + \frac{\partial E_z}{\partial z} = 0, \tag{10}$$

so that,

$$\begin{aligned} E_r(r, z, t) &= \frac{1}{r}\sum_n a_n k_n \int r I_0(K_n r) dr \ \sin k_n z \cos \omega t \\ &= \frac{r}{2}\sum_n a_n k_n \tilde{I}_1(K_n r) \sin k_n z \cos \omega t, \end{aligned} \tag{11}$$

using the fact that $d(xI_1)/dx = xI_0$, and where,

$$\tilde{I}_1(x) = \frac{2I_1(x)}{x} = 1 + \frac{(x/2)^2}{1!2!} + \frac{(x/2)^4}{2!3!} + \cdots \tag{12}$$

The azimuthal component of the magnetic field is obtained from,

$$(\nabla \times \mathbf{E})_\theta = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{1}{c}\frac{\partial B_\theta}{\partial t}, \tag{13}$$

so that,

$$\begin{aligned} B_\theta(r,z,t) &= \frac{c}{\omega}\sum_n a_n \left(\frac{dI_0(K_n r)}{dr} - \frac{k_n^2 r}{2}\tilde{I}_1(K_n r)\right)\cos k_n z \sin\omega t \\ &= \frac{\pi r}{\lambda}\sum_n a_n \tilde{I}_1(K_n r)\cos k_n z \sin\omega t, \end{aligned} \tag{14}$$

using the fact that $I_0'(x) = I_1(x)$.

We desire that the transverse fields E_r and B_θ vary linearly with r. According to (11-12) and (14), this requires that $K_n = 0$. The simplest choice is $n = 1$, $n_0 = 1$, so that $k_n = \pi/2d$ and $d = \lambda/4$. The fields are,

$$E_z = E_0 \cos\frac{\pi z}{2d}\cos\omega t, \tag{15}$$

$$E_r = \frac{\pi r}{4d}E_0 \sin\frac{\pi z}{2d}\cos\omega t, \tag{16}$$

$$B_\theta = \frac{\pi r}{4d}E_0 \cos\frac{\pi z}{2d}\sin\omega t. \tag{17}$$

The cavity length is $2d = \lambda/2$, and E_z vanishes on axis at the ends of the cavity. This configuration is called the π mode in accelerator physics. Since $E_r(z = \pm d) \neq 0$, this mode cannot exist in a structure with conducting walls at the planes $z = \pm d$; apertures are required.

The electric field is perpendicular to the walls of a perfectly conducting cavity. Expressing the shape of the walls as $r(z)$, we then have,

$$\frac{dr}{dz} = -\frac{E_z}{E_r} = -\frac{4d}{\pi r}\cot\frac{\pi z}{wd}, \tag{18}$$

which integrates to the form,

$$r^2 = b^2 - \left(\frac{4d}{\pi}\right)^2 \ln\left|\sin\frac{\pi z}{2d}\right|, \tag{19}$$

where b is the radius of the apertures at $z = \pm d$. Near $z = \pm d$, the profile is a hyperbola. Since $r \to \infty$ as $z \to 0$, no real cavity can support the idealized fields (15-17). However, a cavity with maximum radius $a = 0.4d$ has a Fourier expansion (9) where $a_2 = 0.15a_1$ [1], so the fields can be a good approximation to (15-17) in real devices.

We can obtain additional formal solutions in which $K_n = 0$ for any value of n, and for n_0 either 1 or 2. However, these solutions are not really distinct from (15-17), but are simply the result of combining any number of $\lambda/2$ cells into a larger structure. Such multicell π-mode structures are difficult to operate in practice, because the strong coupling of the fields from one cell to the next makes the useful range of drive frequencies extremely narrow. The main application of π-mode cavities is for so-called rf guns, in which a half cell has a surface at $z \approx 0$ suitable for laser-induced photoemission of electrons, which are then accelerated further in one or a few more subsequent cells [1].

References

[1] K.T. McDonald, *Design of the Laser-Driven RF Electron Gun for the BNL Accelerator Test Facility*, IEEE Trans. Electron Devices, **35**, 2052 (1988),
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