Radiofrequency Quadrupole

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1 The Problem

A radiofrequency quadrupole (RFQ) is a device for focussing beams of charged particles. The electric field in this device can be approximated as that derived from the quasistatic potential,

$$\phi(x, y, t) = \frac{E_0}{2d}(y^2 - x^2)\sin\omega t, \tag{1}$$

where d is a length and ω is the frequency of the field. The magnetic field is ignored in this approximation. While the approximate fields do not satisfy Maxwell's equations, there is little error for $|x|$, $|y| \ll \lambda$, the wavelength of the radiofrequency waves.

Deduce the equations of motion for a particle of charge e and mass m in the radiofrequency quadrupole. Consider solutions of the form,

$$x(t) = f(t) + g(t)\sin\omega t \tag{2}$$

where $g \ll f$ and both f and g are slowly varying compared to $\sin\omega t$. The parameters may be assumed to satisfy the conditions that such solutions exist.

Complete the solution for the particular case that,

$$\begin{aligned} x(0) &= 0, \qquad \dot{x}(0) = v_0\theta_0, & (3)\\ y(0) &= 0, \qquad \dot{y}(0) = 0, & (4)\\ z(0) &= 0, \qquad \dot{z}(0) = v_0, & (5) \end{aligned}$$

with $\theta_0 \ll 1$. At what distance along the z-axis is the first image of the beam "spot", *i.e.*, where the initially diverging beam is brought back to the z-axis?

2 Solution

This problem was abstracted from [1].

The electric field in the RFQ can be obtained from the potential via $\mathbf{E} = -\nabla\phi$, so,

$$\begin{aligned} E_x &= \frac{x}{d}E_0\sin\omega t, & (6)\\ E_y &= -\frac{y}{d}E_0\sin\omega t. & (7) \end{aligned}$$

The equations of motion are,

$$\begin{aligned} \ddot{x} &= \frac{x}{d}\frac{eE_0}{m}\sin\omega t, & (8)\\ \ddot{y} &= -\frac{y}{d}\frac{eE_0}{m}\sin\omega t, & (9)\\ \ddot{z} &= 0. & (10) \end{aligned}$$

Then,

$$z(t) = z_0 + v_{0z}t = v_0 t \tag{11}$$

for the particular case specified.

For the x motion, we consider the form (2),

$$\dot{x} = \dot{f} + \dot{g}\sin\omega t + \omega g\cos\omega t, \tag{12}$$

$$\ddot{x} = \ddot{f} + \ddot{g}\sin\omega t + 2\omega\dot{g}\cos\omega t - \omega^2 g\sin\omega t. \tag{13}$$

The x equation of motion now yields,

$$\ddot{f} + 2\omega\dot{g}\cos\omega t = \left[-\ddot{g} + \omega^2 g + \frac{f + g\sin\omega t}{d}\frac{eE_0}{m}\right]\sin\omega t. \tag{14}$$

Since g is both small and slowly varying by hypothesis, we neglect the terms involving $\dot{g}$ and $\ddot{g}$, leaving,

$$\ddot{f} \approx \left[\omega^2 g + \frac{f}{d}\frac{eE_0}{m}\right]\sin\omega t + \frac{g}{d}\frac{eE_0}{m}\sin^2\omega t. \tag{15}$$

In this, the coefficent of the rapidly varying term $\sin\omega t$ should vanish, and $\ddot{f}$ should be the average of the term in $\sin^2\omega t$. The first condition tells us that,

$$g = -\frac{eE_0}{m\omega^2 d}f, \tag{16}$$

which combines with the (averaged) second condtion to give a differential equation for f,

$$\ddot{f} = -\frac{1}{2}\left(\frac{eE_0}{m\omega d}\right)^2 f. \tag{17}$$

Thus,

$$f \approx A\cos\Omega t + B\sin\Omega t, \qquad \text{where} \qquad \Omega = \frac{eE_0}{\sqrt{2}m\omega d}. \tag{18}$$

Together we have,

$$x(t) \approx (A\cos\Omega t + B\sin\Omega t)\left(1 - \frac{eE_0}{m\omega^2 d}\sin\omega t\right). \tag{19}$$

The particular initial conditions (3-5) are satisifed by,

$$x(t) \approx \frac{v_0\theta_0}{\Omega}\sin\Omega t\left(1 - \frac{eE_0}{m\omega^2 d}\sin\omega t\right). \tag{20}$$

For this to be consistent we must have that,

$$\frac{eE_0}{m\omega^2 d} \ll 1. \tag{21}$$

Then, the beam returns to the z-axis at time $t = \pi/\Omega$, corresponding to distance $z = \pi v_0/\Omega$.

The argument is similar for the y motion. The opposite sign of the electric field leads to,

$$g = +\frac{eE_0}{m\omega^2 d}f, \tag{22}$$

and so,

$$y(t) \approx (C\cos\Omega t + D\sin\Omega t)\left(1 + \frac{eE_0}{m\omega^2 d}\sin\omega t\right). \tag{23}$$

The particular initial conditions (3-5), however, require that both C and D vanish.

Experts will recognize that the dimensionless quantity,

$$\eta \equiv \frac{eE_0}{m\omega c}, \tag{24}$$

where c is the speed of light, is a useful invariant of the field. In terms of this invariant the condition of validity of the solution is,

$$\eta \frac{\lambda}{2\pi d} \ll 1. \tag{25}$$

If d is a characteristic aperture of the RFQ, we earlier required that $\lambda \gg d$ so the quasistatic approximation to the fields would be valid. Hence, the invariant field strength η cannot be too large in the RFQ.

The physical meaning of the invariant η is that it is the ratio of the energy gain over distance $\lambda/2\pi$ to the electron rest energy mc^2,

$$\eta = \frac{eE_0}{m\omega c} = \frac{eE_0\lambda/2\pi}{mc^2}. \tag{26}$$

Thus, the RFQ should not impart relativistic transverse motion to the particles if it is to function as described above.

References

[1] T.P. Wangler, *Strong focusing and the radiofrequency quadrupole accelerator*, Am. J. Phys. **64**, 177 (1996), http://kirkmcd.princeton.edu/examples/EM/wangler_ajp_64_177_96.pdf