Energy Balance in an Electrostatic Accelerator

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1 Problem

The principle of an electrostatic accelerator is that when a charge e escapes from a charged plane which supports a uniform electric field of strength E_0, with $eE_0 > 0$, then the charge gains energy $eE_0 d$ as it moves distance d from the plane. Where does this energy come from?

Show that the mechanical (kinetic) energy gain of the charge is balanced by the decrease in the electrostatic field energy of the system.

You may suppose that radiated energy is negligible, and that the velocity v of the charge is always small compared to the speed c of light in vacuum.

2 Solution

For simplicity, we model the electrostatic accelerator as a single plane of fixed charge density, at $z = 0$, that supports a uniform electric field $\mathbf{E}_0 = E_0\,\hat{\mathbf{z}}$ for $z > 0$ (and $\mathbf{E}_0 = -E_0\,\hat{\mathbf{z}}$ for $z < 0$). The more realistic, but more intricate, case of a conducting plane is discussed in the Appendix.

When the charge has reached distance d from the plane (with velocity $v \ll c$), the electric field $\mathbf{E}_e$ at an arbitrary point $\mathbf{r}$ due to the charge e is approximately its static value,[1]

$$\mathbf{E}_e(\mathbf{r}) \approx e\frac{\mathbf{r}'}{r'^3}\,, \tag{1}$$

in Gaussian units, with $\mathbf{r}' = \mathbf{r} - d\,\hat{\mathbf{z}}$, and $r'^2 = r^2 + (z-d)^2$ in a cylindrical coordinate system (r, θ, z) where the charge was initially at rest at the origin.

The part of the electrostatic field energy that varies with the position of the charge is the interaction term,

$$\begin{aligned}
U_{\rm int} &= \int \frac{\mathbf{E}_0 \cdot \mathbf{E}_e}{4\pi} d\mathrm{Vol} \\
&= \frac{eE_0}{4\pi}\int_0^\infty dz \int_0^\infty \pi\, dr^2 \frac{z-d}{[r^2+(z-d)^2]^{3/2}} - \frac{eE_0}{4\pi}\int_{-\infty}^0 dz \int_0^\infty \pi\, dr^2 \frac{z-d}{[r^2+(z-d)^2]^{3/2}} \\
&= \frac{eE_0}{4\pi}\int_0^\infty dz \int_0^\infty \pi\, dr^2 \left(\frac{z-d}{[r^2+(z-d)^2]^{3/2}} - \frac{z+d}{[r^2+(z+d)^2]^{3/2}}\right) \\
&= \frac{eE_0}{4}\int_0^\infty dz \left(\left\{\begin{array}{ll} 2 & \text{if } z > d \\ -2 & \text{if } z < d \end{array}\right\} - 2\right) = -eE_0\int_0^d dz = -eE_0 d.
\end{aligned} \tag{2}$$

[1]The (retarded) electric field to order $1/c^2$ was expressed in terms of present quantities by Darwin [1], as reviewed in sec. 2.2.1 of [2].

Meanwhile, the kinetic energy of the charge has increased by,

$$\Delta\mathrm{KE} = \int_0^d \mathbf{F}\cdot d\mathbf{x} = \int_0^d eE_0\,dz = eE_0 d, \tag{3}$$

which is equal and opposite to the change (2) in the interaction field energy.[2]

A pre-Maxwellian view is that the potential energy eV of the charge decreased while its kinetic energy increased. However, this is more of a mathematical accounting than a physical explanation, in that the potential energy has no physical location, and the scalar potential V is not gauge invariant.[3] The spirit of the field theory of Faraday and Maxwell [5] is that the electromagnetic field is a dynamical (but not "mechanical") entity which carries energy, and in general, momentum and angular momentum.

2.1 Energy Flow

The flow of electromagnetic energy in the system is described by the Poynting vector [6],

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{4}$$

where in the present example the quasistatic magnetic field of the slow-moving charge is,

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mathbf{v}}{c}\times e\frac{\mathbf{r}'}{r'^3} = \frac{ev}{c}\,\hat{\mathbf{z}}\times\frac{\mathbf{r}_\perp}{r'^3}\,, \tag{5}$$

noting that $\mathbf{r}'_\perp = \mathbf{r}_\perp = \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{z}})\,\hat{\mathbf{z}}$. Of course, the electric field is $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + \mathbf{E}_e \equiv (\pm E_0 + E_{e,z})\,\hat{\mathbf{z}} + \mathbf{E}_{e,\perp}$, where the minus sign holds for $z < 0$. The Poynting vector is then,

$$\mathbf{S}(\mathbf{r}) = \frac{c}{4\pi}\left((\pm E_0 + E_{e,z})\,\hat{\mathbf{z}} + \frac{e}{r'^3}\mathbf{r}_\perp\right)\times\left(\frac{ev}{c}\,\hat{\mathbf{z}}\times\frac{\mathbf{r}_\perp}{r'^3}\right) = \frac{e^2 v\,\mathbf{r}_\perp^2}{4\pi r'^6}\,\hat{\mathbf{z}} - \frac{ev}{4\pi}(\pm E_0 + E_{e,z})\frac{\mathbf{r}_\perp}{r'^3}\,. \tag{6}$$

Electromagnetic energy flows onto the charge, where it is transformed into kinetic energy at rate,[4]

$$P = \mathbf{F}\cdot\mathbf{v} = evE_{\mathrm{on}\ e} = evE_0, \tag{7}$$

when the charge e is at position $z = d$ with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$.[5]

[2] The self-field energy of the charge increases as the velocity of the charge increases (due to the resulting magnetic field), but we consider this self energy (infinite in case of a point charge) to be "renormalized" into the (relativistic) mass of the charge.

[3] For example, one can choose to use the Gibbs' gauge [3, 4], in which the scalar potential is $V = 0$ everywhere (in contrast to $V = -E_0\,|z| + e/r'$ for the present example in the Coulomb gauge).

[4] While kinetic energy can be called "mechanical", electromagnetic energy should not be called that, and there is no "mechanical" model of the transformation of one type of energy into the other.

[5] In addition, electromagnetic energy flows along with the charge at velocity $\mathbf{v}$, with energy in the region "behind" the charge ($z < d$, where the total field, and field energy density $u \approx E^2/8\pi$, decrease with time) being transported to the region in "front" of the charge ($z > d$, where the total field increases with time). When $E_0 = 0$ and v is constant, the total field energy remains constant in this process. But, for positive E_0, the rearrangement of field energy results in its net decrease, with a corresponding increase in the kinetic energy of the accelerated charge.

2.1.1 Energy Flow onto the Charge

We now verify that the integral of the Poynting vector (6) over a small surface surrounding the charge equals the rate of increase (7) of kinetic energy of the charge.

For this, we consider a small cylindrical "pillbox" of radius a and thickness $2b$ centered on the charge. The Poynting flux Φ that enters the surface of this "pillbox" is,

$$\begin{aligned}\Phi &= \int_0^a S_z(r, z = d-b)\, 2\pi r\, dr - \int_0^a S_z(r, z = d+b)\, 2\pi r\, dr - \int_{-b}^b S_\perp(a, z = d+z')\, 2\pi a\, dz' \\ &= \frac{e^2 v}{2}\int_0^a \frac{r^3\, dr}{[r^2+b^2]^3} - \frac{e^2 v}{2}\int_0^a \frac{r^3\, dr}{[r^2+b^2]^3} + \frac{eav}{2}\int_{-b}^b \left(E_0 + \frac{e(z'-d)}{[a^2+z'^2]^{3/2}}\right)\frac{a}{[a^2+z'^2]^{3/2}}\, dz' \\ &= \frac{ea^2 v E_0}{2}\int_{-b}^b \frac{dz'}{[a^2+z'^2]^{3/2}} + \frac{e^2a^2v}{2}\int_{-b}^b \frac{z'\ dz'}{[a^2+z'^2]^3} - \frac{e^2a^2 d\, v}{2}\int_{-b}^b \frac{dz'}{[a^2+z'^2]^3}\,. \end{aligned} \tag{8}$$

The first integral in the last line of eq. (8) has the value, using Dwight 200.03,

$$\frac{ea^2 v E_0}{2}\left(\frac{2b}{a^2\sqrt{a^2+b^2}}\right), \tag{9}$$

which goes to evE_0 as a and b go to zero, if we suppose that $a \ll b$ always.

The second integral is odd in z', and so vanishes.

The third integral is ill defined as a and b go to zero, but it is physically consistent to take this integral to be zero in the limit that $b = 0$.

Then, the Poynting flux that flows onto the charge is, $\Phi = evE_0$, in agreement with the rate (7) of increase of the kinetic energy of the charge.

The view of Maxwell and Poynting of the charge being accelerated by the nominally static field $E_0\,\hat{\mathbf{z}}$ is that the energy gained by the charge comes from a reduction of the electromagnetic field energy surrounding the charge, and this energy flows out of the field onto the charge.[6]

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[6]Various alternatives to the Poynting vector have been proposed, but so far none has supplanted the form (4). For a review, see [14]. A form that associates the flow of electromagnetic energy with the flow of electric current was advocated by Livens in 1917, sec. 4 of [15] (see also secs. 627-628, pp. 555-556 of his textbook [16], and sec. 229, pp. 242-244 of the 2nd edition [17]),

$$\mathbf{S}^{(\mathrm{Livens})} = V\mathbf{J}_{\mathrm{total}}, \tag{10}$$

where V is the electrical scalar potential in the Coulomb gauge, and $\mathbf{J}_{\mathrm{total}} = \mathbf{J}_{\mathrm{charges}} + (1/4\pi)\partial\mathbf{D}/\partial t$ is the total current density, including the "displacement current" (introduced by Maxwell in eq. (112) of [18]).

In the present example, the scalar potential is, for $z > 0$, $V(\mathbf{r}) = -E_0\,z + e/r'$, and the total current density is, noting that $\mathbf{D} = \mathbf{E}$ in the present example, and that $d\mathbf{r}'/dt = -\mathbf{v}$,

$$\mathbf{J}_{\mathrm{total}}(\mathbf{r}) = e\mathbf{v}\,\delta^3(\mathbf{r} - d\,\hat{\mathbf{z}}) + \frac{1}{4\pi}\frac{d}{dt}\frac{e\,\mathbf{r}'}{r'^3} = e\mathbf{v}\,\delta^3(\mathbf{r} - d\,\hat{\mathbf{z}}) - \frac{e\,\mathbf{v}}{4\pi r'^3} + \frac{3ev(z-d)\,\mathbf{r}'}{4\pi r'^5}\,. \tag{11}$$

Hence, the Poynting vector according to Livens is, for $z > 0$,

$$\mathbf{S}^{(\mathrm{Livens})}(\mathbf{r}) = \left(eE_0\,z - \frac{e^2}{r'}\right)\left(-\mathbf{v}\,\delta^3(\mathbf{r} - d\,\hat{\mathbf{z}}) + \frac{\mathbf{v}}{4\pi r'^3} - \frac{3v(z-d)\,\mathbf{r}'}{4\pi r'^5}\right). \tag{12}$$

This vector, with a delta-function at the position of the charge, does not represent a smooth flow of energy. Rather, it seems to imply that energy appears at the charge without passing through the surrounding electromagnetic field, contrary to the vision of Poynting.

2.2 Momentum

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When the electron has reached $z = d$, and has kinetic energy $T = eEd$, its mechanical momentum is, for $v \ll c$,

$$\mathbf{P}_{\rm mech} = \sqrt{2mT}\,\hat{\mathbf{z}} = \sqrt{2meEd}\,\hat{\mathbf{z}}. \qquad (13)$$

For momentum to be conserved, the mechanical momentum (13) should be equal and opposite to some other momentum. The static electric field has no field momentum, and the momentum in the field of a moving charge diverges as the size of the charge approaches zero. So, we consider interaction field momentum,

$$\mathbf{P}_{\rm int} = \frac{c}{4\pi}\int \mathbf{E}\times\mathbf{B}_e\,d{\rm Vol}. \qquad (14)$$

However, since $\mathbf{E} = E\,\hat{\mathbf{z}}$ and $\mathbf{B}_e$ is azimuthal, the integrand is radial in a cylindrical coordinate system and is axially symmetric, such that the interaction field momentum (14) is zero.

It could be that the mechanical momentum of the accelerated electron comes from its (divergent) self field momentum. That is, the "mechanical" and "electromagnetic" momenta of the electron cannot be crisply distinguished, as perhaps first discussed by J.J. Thomson [7]. However, for acceleration in a uniform, constant electric field, the electromagnetic self force (radiation reaction) vanishes.[7] Since it is reasonable to identify the electromagnetic self force with $-d\mathbf{P}, {\rm EM}, {\rm self}/dt$,[8] we cannot explain the mechanical momentum (in this case) as coming from the momentum of the self fields.

Rather, we note that the accelerated charge exerts a force on the source charges of the uniform field $\mathbf{E}$, and if these charges reside on a free body, that body would take on a mechanical momentum opposite to that of the accelerated charge.

3 Comments

In a practical "electrostatic" accelerator, electrons (of charge $-e$) are freed from rest on an electrode at potential $-V$ and emerge with energy eV into a region of zero potential beyond the ground electrode. However, the electrons can not be brought to the negative electrode from a region of zero potential by purely electrostatic forces, since these forces oppose the desired transport. An "electrostatic" accelerator must have an essential component (such as a battery) that has a nonelectrostatic force that can do work against the electrostatic field while moving the electron from potential 0, so as to put the charge at rest at potential $-V$ prior to acceleration.

The nonelectrostatic component also provides the energy that is stored at potential energy eV when an electron has been placed on the negative electrode. Can we say more precisely where this potential energy is stored?

[7] See, for example, sec. 2.1 of [8].

[8] See [9] and Appendix B of [10].

In a pair of interesting papers [11, 12], Leon Brillouin discussed the theme of mass renormalization in classical electrodynamics.[9] He argued that the electrostatic potential energy eV contributes an amount eV/c^2 to the total "relativistic mass" of the charge, according to Einstein's insight that $E = mc^2$. In his second paper [12], Brillouin considered the example of an electrostatic accelerator (where the potential energy eV can exceed the rest energy m_0c^2 of an electron). If Brillouin were correct, the relativistic mass of an electron would be greater than γm_0 when it has been accelerated to velocity $v = c\sqrt{1 - 1/\gamma^2}$ and is still within the region of nonzero electric potential. Therefore, the magnitude of the acceleration would be less than if the relativistic mass of the electron were γm_0, and the acceleration would be slower than expected is the usual analysis. Once the electron has left the accelerator and is in a region of zero potential, its velocity obeys the usual relation $\gamma m_0c^2 = eV$.

But, is Brillouin correct? He implied that the Lorentz force law is not,

$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m_0 \mathbf{v})}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{(Lorentz)}, \tag{15}$$

but rather,

$$\frac{d\mathbf{p}'}{dt} = \frac{d(\gamma m_0 + eV/c^2)\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{(Brillouin)}. \tag{16}$$

The great success in the application of the usual Lorentz force law to relativistic particle accelerators argues against the validity of Brillouin's proposed classical mass renormalization.

Furthermore, the present problem shows that there is a decrease in the interaction field energy as the electron is accelerated. In the spirit of Brillouin, should we consider this energy to contribute to the mass of the electron? If so, the electron could acquire a negative mass while still inside an accelerator that uses sufficiently high voltage, which seems preposterous.

A Appendix: Conducting Plane at $z = 0$

We now consider a model of an electrostatic accelerator as a conducting plane, at $z = 0$, which supports electric field $\mathbf{E}_0 = E_0\,\hat{\mathbf{z}}$ for $z > 0$.[10]

[9]Brillouin's consideration of classical mass renormalization was closely, but apparently independently, followed by a different vision. Namely, that if a charge e is in potential V in such a way that its total energy is the same as if V were zero, then the mechanical mass of the charge is reduced by amount eV/c^2. If in addition, the charge has a velocity $\mathbf{v}$, and hence a mechanical momentum, then that momentum is lower than when $V = 0$ by $e\mathbf{v}V/c^2$. This phenomenon is sometimes called "hidden momentum" [13].

[10]In general, the uniform electric field $\mathbf{E}_0$ terminates at a planar electrode at $z = D$. In this case, the electric field E_e associated with the charge e at $z = d < D$ can be deduced from an infinite set of image charges; $+e$ at $z = 2nD + d$ and $-e$ at $z = 2nD - d$, where $n = 0, \pm1, \pm2, ...$ To calculate the interaction energy $U_{\rm int}(\mathbf{E}_0, \mathbf{E}_e)$ it is convenient to group these charges into pairs whose positions are symmetric about $z = 0$. Pairs that have charge $+e$ at $z > 0$ have z coordinates $\pm(2nD + d)$ where now the integer n takes on only the values 0,1,2,3,..., while pairs that have charge $-e$ at $z > 0$ have z coordinates $\pm(2mD - d)$ where $m = 1, 2, 3, ...$ The interaction energy for each pair can be calculated as in eq. (18), with the integration in z only between 0 and D. For $n = 0$ the integral is $-eE_0d$ as above. For pairs with $+e$ at $z > 0$ and $n > 0$ the integral is $-eE_0D$, and for pairs with $-e$ at $z > 0$ (and $m > 0$) the integral is eE_0D. The interaction energies cancel for each pair of pairs with $n = m$, and the total interaction energy remains $U_{\rm int}(\mathbf{E}_0, \mathbf{E}_e) = -eE_0d$.

Another issue with a second conducting plane is that when the accelerated charge passes into/through it, transition radiation is generated in the space between two planes. The energy of this radiation comes from a further reduction of the quasistatic field energy between the planes.

Once the charge has reached distance d from the plane, the static electric field $\mathbf{E}_e$ at an arbitrary point $\mathbf{r}$ due to the charge can be calculated (in Gaussian units) by summing the field of the charge plus its image charge,

$$\mathbf{E}_e(\mathbf{r}, d) \equiv \mathbf{E}_1 + \mathbf{E}_2 = e\frac{\mathbf{r}_1}{r_1^3} - e\frac{\mathbf{r}_2}{r_2^3}, \tag{17}$$

where $\mathbf{r}_{1,2} = \mathbf{r} \mp d\,\hat{\mathbf{z}}$ points from the charge e (image, $-e$) to the observation point $\mathbf{r}$, as illustrated in Fig. 1. The total electric field is then $E_0\,\hat{\mathbf{z}} + \mathbf{E}_e$.

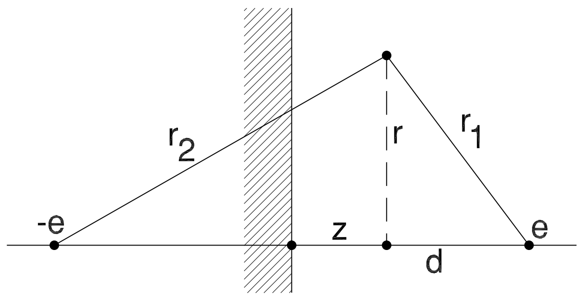

Figure 1: The charge e and its image charge $-e$ at positions $(r, \theta, z) = (0, 0, \pm d)$ with respect to a conducting plane at $z = 0$. Vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed from the charges to the observation point $(r, 0, z)$.

We again use a cylindrical coordinate system, where the observation point is $\mathbf{r} = (r, \theta, z) = (r, 0, z)$, and the charge is at $(0, 0, d)$. Then, $r_{1,2}^2 = r^2 + (z \mp d)^2$.

In a classical model, an electric charge could never leave a conducting plane, as it has infinite "binding energy" with its image charge. We could suppose that the charge starts from a location $d_0 > 0$ with a nonzero initial velocity v_0 at time $t = 0$, sufficient to permit it to reach $z = d$ at some later time.

The part of the electrostatic field energy that varies with the position of the charge is the interaction term,

$$\begin{aligned}
U_{\rm int}(d) &= \int \frac{E_0\hat{\mathbf{z}}\cdot\mathbf{E}_e}{4\pi} d\mathrm{Vol} + \int \frac{\mathbf{E}_1\cdot\mathbf{E}_2}{4\pi} d\mathrm{Vol} \\
&= \frac{eE_0}{4\pi}\int_0^\infty dz \int_0^\infty \pi\, dr^2 \left(\frac{z-d}{[r^2+(z-d)^2]^{3/2}} - \frac{z+d}{[r^2+(z+d)^2]^{3/2}}\right) \\
&\quad -\frac{e^2}{4\pi}\int_0^\infty dz \int_0^\infty \pi\, dr^2 \frac{r^2+z^2-d^2}{[r^2+(z-d)^2]^{3/2}\,[r^2+(z+d)^2]^{3/2}} \\
&= \frac{eE_0}{4}\int_0^\infty dz \left(\left\{\begin{array}{ll} 2 & \text{if } z > d \\ -2 & \text{if } z < d \end{array}\right\} - 2\right) - \frac{e^2}{4}\int_0^\infty dz \left\{\begin{array}{ll} \frac{1}{z^2} & \text{if } z > d \\ 0 & \text{if } z < d \end{array}\right\} \\
&= -eE_0\int_0^d dz - \frac{e^2}{4}\int_d^\infty \frac{dz}{z^2} \\
&= -eE_0 d - \frac{e^2}{4d},
\end{aligned} \tag{18}$$

where we used Wolfram Alpha to evaluate the second integral [19].

The initial interaction energy is then,

$$U_{\rm int}(d_0) = -eE_0 d_0 - \frac{e^2}{4d_0}\,, \tag{19}$$

so the change in the interaction field energy as the charge moves from d_0 to d is,

$$\Delta U_{\rm int} = U_{\rm int}(d) - U_{\rm int}(d_0) = -eE_0(d-d_0) - \frac{e^2}{4d} + \frac{e^2}{4d_0}\,. \tag{20}$$

Meanwhile, the kinetic energy of the charge has increased by,

$$\Delta{\rm KE} = \int_{d_0}^{d} \mathbf{F}\cdot d\mathbf{x} = \int_{d_0}^{d} e\left(E_0 - \frac{e}{4z^2}\right)\,dz = eE_0(d-d_0) + \frac{e^2}{4d} - \frac{e^2}{4d_0}\,, \tag{21}$$

which is equal and opposite to the change (20) in the interaction field energy.

A.1 Energy Flow

In the example with a conducting plane, the quasistatic magnetic field for low velocities $\mathbf{v} = v\,\hat{\mathbf{z}}$ is that due to the moving charge and its image,

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mathbf{v}}{c}\times e\frac{\mathbf{r}_1}{r_1^3} + \frac{-\mathbf{v}}{c}\times -e\frac{\mathbf{r}_2}{r_2^3} = \frac{ev}{c}\,\hat{\mathbf{z}}\times\left(\frac{\mathbf{r}_1}{r_1^3} + \frac{\mathbf{r}_2}{r_2^3}\right) = \frac{ev}{c}\,\hat{\mathbf{z}}\times\left(\frac{\mathbf{r}_{1,\perp}}{r_1^3} + \frac{\mathbf{r}_{2,\perp}}{r_2^3}\right). \tag{22}$$

Again, the electric field is $\mathbf{E}(\mathbf{r}) = E_0\,\hat{\mathbf{z}} + \mathbf{E}_e \equiv (E_0 + E_{e,z})\,\hat{\mathbf{z}} + \mathbf{E}_{e,\perp}$. The Poynting vector is then, noting that $\mathbf{r}_{1,\perp} = \mathbf{r}_{2,\perp} = \mathbf{r}_\perp = \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{z}})\,\hat{\mathbf{z}}$,

$$\begin{aligned}
\mathbf{S}(\mathbf{r}) &= \frac{c}{4\pi}\left[(E_0 + E_{e,z})\,\hat{\mathbf{z}} + e\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)\mathbf{r}_\perp\right]\times\left[\frac{ev}{c}\,\hat{\mathbf{z}}\times\left(\frac{1}{r_1^3} + \frac{1}{r_2^3}\right)\mathbf{r}_\perp\right] \\
&= \frac{e^2 v\,\mathbf{r}_\perp^2}{4\pi}\left(\frac{1}{r_1^6} - \frac{1}{r_2^6}\right)\hat{\mathbf{z}} - \frac{ev}{4\pi}(E_0 + E_{e,z})\left(\frac{1}{r_1^3} + \frac{1}{r_2^3}\right)\mathbf{r}_\perp.
\end{aligned} \tag{23}$$

Electromagnetic energy flows onto the charge where it is transformed into kinetic energy at rate,

$$P = \mathbf{F}\cdot\mathbf{v} = evE_{{\rm on}\ e} = ev\left(E_0 - \frac{e}{4d^2}\right), \tag{24}$$

when the charge e is at position $z = d$ with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$. For $P > 0$, the rearrangement of field energy results in its decrease, with a corresponding increase in the kinetic energy of the accelerated charge.

We now verify that the integral of the Poynting vector (23) over a small surface surrounding the charge equals the rate of increase (24) of kinetic energy of the charge.

For this, we consider a small cylindrical "pillbox" of radius a and thickness $2b$ centered on the charge. The Poynting flux Φ that enters the surface of this "pillbox" is,

$$
\begin{aligned}
\Phi &= \int_0^a S_z(r, z=d-b)\, 2\pi r\, dr - \int_0^a S_z(r, z=d+b)\, 2\pi r\, dr - \int_{-b}^{b} S_\perp(a, z=d+z')\, 2\pi a\, dz' \\
&= \frac{e^2 v}{2} \int_0^a \left(\frac{1}{[r^2+b^2]^3} - \frac{1}{[r^2+(2d-b)^2]^3} \right) r^3\, dr \\
&- \frac{e^2 v}{2} \int_0^a \left(\frac{1}{[r^2+b^2]^3} - \frac{1}{[r^2+(2d+b)^2]^3} \right) r^3\, dr \\
&+ \frac{eav}{2} \int_{-b}^{b} \left(E_0 + \frac{e(z'-d)}{[a^2+z'^2]^{3/2}} - \frac{e(z'+2d)}{[a^2+(2d+z')^2]^{3/2}} \right) \left(\frac{a}{[a^2+z'^2]^{3/2}} + \frac{a}{[a^2+(2d+z')^2]^{3/2}} \right) dz' \\
&= \frac{e^2 v}{2} \int_0^a \left(\frac{1}{[r^2+(2d+b)^2]^3} - \frac{1}{[r^2+(2d-b)^2]^3} \right) r^3\, dr \\
&+ \frac{ea^2 v E_0}{2} \int_{-b}^{b} \left(\frac{1}{[a^2+z'^2]^{3/2}} + \frac{1}{[a^2+(2d+z')^2]^{3/2}} \right) dz' \\
&+ \frac{e^2 a^2 v}{2} \int_{-b}^{b} \left(\frac{1}{[a^2+z'^2]^3} - \frac{1}{[a^2+(2d+z')^2]^3} \right) z'\, dz' \\
&- \frac{e^2 a^2 d\, v}{2} \int_{-b}^{b} \left(\frac{1}{[a^2+z'^2]^3} + \frac{2}{[a^2+(2d+z')^2]^3} + \frac{3}{[a^2+z'^2]^{3/2}\,[a^2+(2d+z')^2]^{3/2}} \right) dz'.
\end{aligned} \tag{25}
$$

The first integral in the final form of eq. (25) goes to zero as b goes to zero.

The second integral, involving E_0, has the value, using Dwight 200.03 and 380.003,

$$
\frac{ea^2 v E_0}{2} \left(\frac{2b}{a^2\sqrt{a^2+b^2}} + \frac{4b+8d}{4a^2\sqrt{a^2+(2d+b)^2}} - \frac{-4b+8d}{4a^2\sqrt{a^2+(2d-b)^2}} \right), \tag{26}
$$

which goes to evE_0 as a and b go to zero, if we suppose that $a \ll b$ always.

The third integral is odd in z', and so vanishes.

The fourth integral is complicated. Here, we suppose that as b goes to zero, the contribution of the first two terms in this integral can be neglected, while in the third term we approximate $[a^2+(2d+z')^2]^{3/2}$ by $8d^3$. Then, this integral is approximately,

$$
-\frac{3e^2 a^2 v}{16 d^2} \int_{-b}^{b} \frac{dz'}{(a^2+z'^2)^{3/2}} = -\frac{3e^2 v b}{8d^2\sqrt{a^2+b^2}}, \tag{27}
$$

which goes to $-3e^2 v/8d^2$ as a and b go to zero, again supposing that $a \ll b$ always.

Thus, the Poynting flux onto the charge is, in the above approximations,

$$
\Phi \approx ev \left(E_0 - \frac{3e}{8d^2} \right), \tag{28}
$$

which is essentially the rate (24) of increase of the kinetic energy of the charge, when $E_0 \gg e/d^2$.

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