Radiation from a Superluminal Source

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

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Abstract

The sweep speed of an electron beam across the face of an oscilloscope can exceed the velocity of light, although of course the velocity of the electrons does not. Associated with this possibility there should be a kind of Čerenkov radiation, as if the oscilloscope trace were due to a charge moving with superluminal velocity.

1 Introduction

The possibility of radiation from superluminal sources was first considered by Heaviside in 1888 [1]. He discussed this topic many times over the next 20 years, deriving most of the formalism of what is now called Čerenkov radiation. However, despite being an early proponent of the concept of a velocity-dependent electromagnetic mass, Heaviside never acknowledged the limitation that massive particles must have velocities less than that of light. Consequently many of his pioneering efforts (and those of his immediate followers, Des Coudres [2] and Sommerfeld [3]), were largely ignored, and the realizable case of radiation from a charge with velocity greater than the speed of light in a dielectric medium was discovered independently in an experiment by Čerenkov in 1934 [4].

In an insightful discussion of the theory of Čerenkov radiation, Tamm [5] deduced its close connection with what is now called transition radiation, *i.e.*, radiation emitted by a charge in uniform motion that crosses a boundary between metallic or dielectric media. The present paper was inspired by a work of Bolotovskii and Ginzburg [6] on how aggregates of particles can act to produce motion that has superluminal aspects and that there should be corresponding Čerenkov-like radiation in the case of charged particles. The classic example of aggregate superluminal motion is the velocity of the point of intersection of a pair of scissors whose tips approach one another at a velocity close to that of light.

Here we consider the example of a "sweeping" electron beam in a high-speed analog oscilloscope such as the Tektronix 7104 [7]. In this device the "writing speed", the velocity of the beam spot across the faceplate of the oscilloscope, can exceed the speed of light. The transition radiation emitted by the beam electrons just before they disappear into the faceplate has the character of Čerenkov radiation from the superluminal beam spot, according to the inverse of the argument of Tamm.

2 Model Calculation

As a simple model suppose a line of charge moves in the $-y$ direction with velocity $u \ll c$, where c is the speed of light, but has a slope such that the intercept with the x axis moves with velocity $v > c$. See Figure 1a. If the region $y < 0$ is occupied by, say, a metal, the charges will emit transition radiation as they disappear into the metal's surface. Interference among the radiation from the various charges then leads to a strong peak in the radiation pattern at angle $\cos\theta = c/v$, which is the Čerenkov effect of the superluminal source.

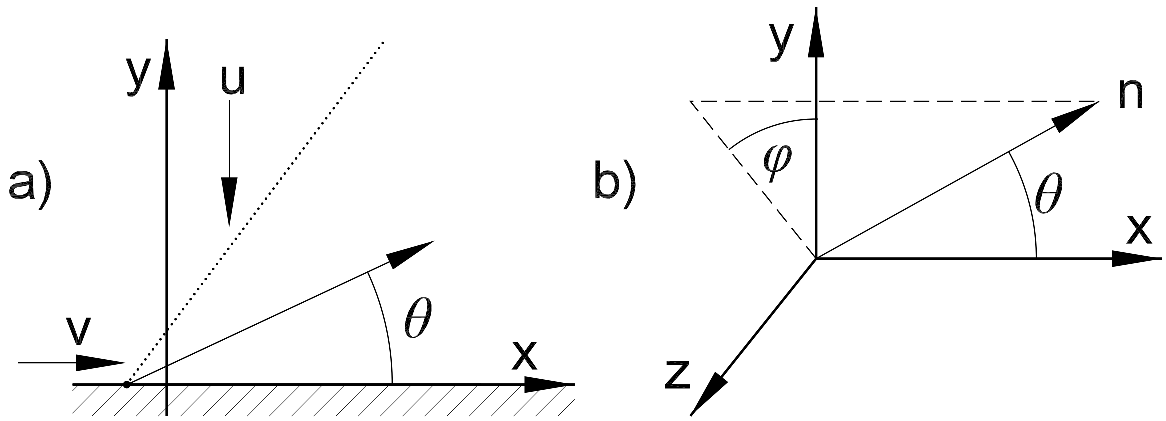

Figure 1: a) A sloping line of charge moves in the $-y$ direction with velocity $v_y = u \ll c$ such that its intercept with the x axis moves with velocity $v_x = v > c$. As the charge disappears into the conductor at $y < 0$ it emits transition radiation. The radiation appears to emanate from a spot moving at superluminal velocity and is concentrated on a cone of angle $\cos^{-1}(c/v)$. b) The angular distribution of the radiation is discussed in a spherical coordinates system about the x axis.

To calculate the radiation spectrum we use equation (14.70) from the textbook of Jackson [8],

$$\frac{dU}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left[\int dt\, d^3r\, \widehat{\mathbf{n}} \times \mathbf{J}(\mathbf{r},t)\, e^{i\omega(t-(\widehat{\mathbf{n}}\cdot\mathbf{r})/c)}\right]^2, \tag{1}$$

where dU is the radiated energy in angular frequency interval $d\omega$ emitting into solid angle $d\Omega$, $\mathbf{J}$ is the source current density, and $\widehat{\mathbf{n}}$ is a unit vector towards the observer.

The line of charge has equation,

$$y = \frac{u}{v}x - ut, \qquad z = 0, \tag{2}$$

so the current density is,

$$\mathbf{J} = -Ne\,\delta(z)\,\delta\left(t - \frac{x}{v} + \frac{y}{u}\right)\widehat{\mathbf{y}}, \tag{3}$$

where N is the number of electrons per unit length intercepting the x axis, and $e < 0$ is the electron's charge.

We also consider the effect of the image current,

$$\mathbf{J}_{\text{image}} = +(-Ne)\,\delta(z)\,\delta\left(t - \frac{x}{v} - \frac{y}{u}\right)\widehat{\mathbf{y}}. \tag{4}$$

We will find that to a good approximation the image current just doubles the amplitude of the radiation. For $u \approx c$ the image current would be related to the retarded fields of the electron beam, but we avoid this complication when $u \ll c$. Note that the true current exists only for $y > 0$, while the image current applies only for $y < 0$.

We integrate using rectangular coordinates, with components of the unit vector $\mathbf{n}$ given by

$$n_x = \cos\theta, \qquad n_y = \sin\theta\cos\phi, \qquad \text{and} \qquad n_z = \sin\theta\sin\phi, \tag{5}$$

as indicated in Fig. 1b. The current impinges only on a length L along the x axis. The integrals are elementary and we find, noting $\omega/c = 2\pi/\lambda$,

$$\frac{dU}{d\omega\, d\Omega} = \frac{e^2 N^2 L^2}{\pi^2 c}\frac{u^2}{c^2}\frac{\cos^2\theta + \sin^2\theta\sin^2\phi}{(1 - \frac{u^2}{c^2}\sin^2\theta\cos^2\phi)^2}\left(\frac{\sin\left[\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)\right]}{\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)}\right)^2. \tag{6}$$

The factor of form $\sin^2\chi/\chi^2$ appears from the x integration, and indicates that this leads to a single-slit interference pattern.

We will only consider the case that $u \ll c$, so from now on we approximate the factor $1 - \frac{u^2}{c^2}\sin^2\theta\cos^2\phi$ by 1.

Upon integration over the azimuthal angle ϕ from $-\pi/2$ to $\pi/2$ the factor $\cos^2\theta + \sin^2\theta\sin^2\phi$ becomes $\frac{\pi}{2}(1 + \cos^2\theta)$.

It is instructive to replace the radiated energy by the number of radiated photons: $dU = \hbar\omega\, dN_\omega$. Thus,

$$\frac{dN_\omega}{d\cos\theta} = \frac{\alpha}{2\pi}\frac{d\omega}{\omega}N^2L^2\frac{u^2}{c^2}(1 + \cos^2\theta)\left(\frac{\sin\left[\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)\right]}{\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)}\right)^2, \tag{7}$$

where $\alpha = e^2/\hbar c \approx 1/137$. This result applies whether $v < c$ or $v > c$. But for $v < c$, the argument $\chi = \frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)$ can never become zero, and the diffraction pattern never achieves a principal maximum. The radiation pattern remains a slightly skewed type of transition radiation. However, for $v > c$ we can have $\chi = 0$, and the radiation pattern has a large spike at angle $\theta_{\check{C}}$ such that,

$$\cos\theta_{\check{C}} = \frac{c}{v}, \tag{8}$$

which we identify with Čerenkov radiation. Of course, the side lobes are still present, but not very prominent.

3 Discussion

The present analysis suggests that Čerenkov radiation is not really distinct from transition radiation, but is rather a special feature of the transition radiation pattern which emerges under certain circumstances. This viewpoint is relevant to Čerenkov radiation in any real device which has a finite path length for the radiating charge. The walls that define the path length are sources of transition radiation, which is always present even when the Čerenkov condition is not satisfied. When the Čerenkov condition is satisfied, the so-called formation length[1] for transition radiation becomes longer than the device, and the Čerenkov radiation can be thought of as an interference effect.

[1] See [9] for a discussion of the formation length and related concepts from a somewhat different perspective than that presented here.

If $L/\lambda \gg 1$, then the radiation pattern is very sharply peaked about the Čerenkov angle, and we may integrate over θ, noting that,

$$d\chi = \frac{\pi L}{\lambda}\, d\cos\theta, \qquad \text{and} \qquad \int_{-\infty}^{\infty} d\chi \frac{\sin^2\chi}{\chi^2} = \pi, \tag{9}$$

to find,

$$dN_\omega \approx \frac{\alpha}{2\pi}(N\lambda)^2 \frac{d\omega}{\omega}\frac{L}{\lambda}\frac{u^2}{c^2}\left(1 + \frac{c^2}{v^2}\right). \tag{10}$$

In this, we have replaced $\cos^2\theta$ by c^2/v^2 in the vicinity of the Čerenkov angle. We have also extended the limits of integration on χ to $[-\infty, \infty]$. This is not a good approximation for $v < c$, in which case $\chi > 0$ always and dN_ω is much less than stated. For $v = c$, the radiation rate is still about one half of the above estimate.

For comparison, the expression for the number of photons radiated in the ordinary Čerenkov effect is,

$$dN_\omega \approx 2\pi\alpha \frac{d\omega}{\omega}\frac{L}{\lambda}\sin^2\theta_{\check{C}}. \tag{11}$$

The ordinary Čerenkov effect vanishes as $\theta^2_{\check{C}}$ near the threshold, but the superluminal effect does not. This is related to the fact that at threshold ordinary Čerenkov radiation is emitted at small angles to the electron's direction, while in the superluminal case the radiation is at right angles to the electron's motion. In this respect the moving spot on an oscilloscope is not fully equivalent to a single charge as the source of the Čerenkov radiation.

In the discussion thus far, we have assumed that the electron beam is well described by a uniform line of charge. In practice the beam would be discrete, with fluctuations in the spacing and energy of the electrons. If these fluctuations were too large we could not expect the transition radiation from the various electrons to superimpose coherently to produce the Čerenkov radiation. Roughly, there would be almost no coherence for wavelengths smaller than the actual spot size of the electron beam at the metal surface. Thus, there would be a cutoff at high frequencies which serves to limit the total radiated energy to a finite amount, whereas the expression derived above is formally divergent. Similarly, the effect will be quite weak unless the beam current is large enough that $N\lambda \gg 1$.

We close with a numerical example inspired by possible experiment. A realistic spot size for the beam is 0.3 mm, so we must detect radiation at longer wavelengths. A convenient choice is $\lambda = 3$ mm, for which commercial microwave receivers exist. The bandwidth of a candidate receiver is $d\omega/\omega = 0.02$, centered at 88 GHz. We take $L = 3$ cm, so $L/\lambda = 10$ and the Čerenkov "cone" will actually be about 5° wide, which happens to match the angular resolution of the microwave receiver. Supposing the electron-beam energy to be 2.5 keV, we would have $u^2/c^2 = 0.01$. The velocity of the moving spot is taken as $v = 1.33c = 4 \times 10^{10}$ cm/sec, so the observation angle is 41°. If the electron beam current were 1 μA, then the number of electrons deposited per cm along the metal surface would be $N \approx 150$, and $N\lambda \approx 45$.

Inserting these parameters into the rate formula (10) we expect about 7×10^{-3} detected photons from a single sweep of the electron beam. This supposes we could collect over all azimuth ϕ, which would require some suitable optics. The electron beam would actually be swept at about 1 GHz, so we could collect about 7×10^6 photons per second. The corresponding signal power would be 2.6×10^{-25} Watts/Hz, whose equivalent noise temperature is about 20 mK. This must be distinguished from the background of thermal radiation, the main source of which would be in the receiver itself, whose noise temperature would be about 100 °K [10]. A lock-in amplifier could be used to extract the weak periodic signal; an integration time of a few minutes of the 1-GHz-repetition-rate signal would suffice, assuming 100% collection efficiency.

Realization of such an experiment with a Tektronix 7104 oscilloscope would require a custom cathode ray tube that permits collection of microwave radiation through a portion of the wall not coated with the usual metallic shielding layer [11].

4 Appendix: Bremsstrahlung

Early reports of observation of transition radiation were considered by skeptics to be due to Bremsstrahlung instead. The distinction in principle is that transition radiation is due to acceleration of charges in a medium in response to the far field of a uniformly moving charge, while Bremsstrahlung is due to the acceleration of the moving charge in the near field of atomic nuclei. In practice, both effects exist, and can be separated by careful experiment.

Is Bremsstrahlung stronger than transition radiation in the example considered here? As shown below, the answer is no, but even if it were we would then expect a Čerenkov-like effect arising from the coherent Bremsstrahlung of the electron beam as it hits the oscilloscope faceplate.

The angular distribution of Bremsstrahlung from a nonrelativistic electron will be $\sin^2\theta$ with θ defined with respect to the direction of motion. The range of a 2.5-keV electron in, say, copper is about 5×10^{-6} cm [12], while the skin depth at 88 GHz is about 2.5×10^{-5} cm. Hence, the copper is essentially transparent to the backward hemisphere of Bremsstrahlung radiation, which would emerge into the same half space as the transition radiation.

The amount of Bremsstrahlung energy dU_B emitted into energy interval dU is just $Y\,dU$ where Y is the so-called Bremsstrahlung yield factor. For 2.5-keV electrons in copper, $Y = 3 \times 10^{-4}$ [12]. The number dN of Bremsstrahlung photons of energy $\hbar\omega$ in a bandwidth $d\omega/\omega$ is then $dN = dU_B/\hbar\omega = Y\,d\omega/\omega$. For the 2% bandwidth of our example, $dN = 6 \times 10^{-6}$ per beam electron. For a 3-cm-long target region there would be 500 beam electrons per sweep of the oscilloscope, for a total of 3×10^{-4} Bremsstrahlung photons into a 2%-bandwidth about 88 GHz. Half of these would emerge from the faceplate as a background to the 7×10^{-3} transition-radiation photons per sweep. Altogether, the Bremsstrahlung contribution would be about 1/50 of the transition-radiation signal in the proposed experiment.

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