# A Maxwellian Perspective on Particle Acceleration

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## Abstract

In a Maxwellian view, an accelerator of charged particles converts electromagnetic field energy into mechanical energy of the particles. A conventional accelerator based on a resonant cavity emphasizes interference between the drive fields and the the spontaneous radiation fields (those fields radiated when charged particles pass through an otherwise empty cavity). In this case the interference term, and hence the particles' energy gain, is linear with the drive field. Particles can also extract positive mechanical energy from a drive wave in free space, where the corresponding decrease in field energy occurs due to interference between the drive fields and the radiated fields of the charged particles as a result of their motion in the drive field. This viewpoint leads to the conclusion that a focused laser pulse in vacuum (far from any matter) can impart an energy gain $\Delta U_{\max} \approx \gamma_0 \eta^2 mc^2$ to a free electron of initial Lorentz factor $\gamma_0$. This gain is quadratic in the strength $\eta = eE_0/m\omega c$ of the laser field. A condition for maximal energy gain is that the Rayleigh range of the laser focus be equal to the formation length $\gamma_0^2 \eta^2 \lambda\!\!\!^-$, and consequently the maximal accelerating gradient varies inversely with electron energy. The maximal energy gain can, in principle, be achieved with linearly or circularly polarized laser beams with optical elements far from the laser focus, or with an axicon beam where the optical elements form a laser cavity of length approximately equal to the Rayleigh range.

# 1 Introduction

The development of tabletop teraWatt lasers has renewed interest in the use of lasers to transfer energy to(accelerate) beams of electrons, as was first proposed by Shimoda in 1962 [1]. In particular, the highest electric fields can be obtained by short laser pulses focused in vacuum, so the question arises as to what extent vacuum laser acceleration of free electrons is possible. This topic has been a subject of debate for some years [2], and recently the vacuum laser acceleration of electrons to 1 MeV has been reported [3]. Here, we give a qualitative but general argument that identifies the dependence of energy transfer from a laser to an electron on relevant dimensional quantities. This approach complements numerical analyses that are typically based on approximate solutions to Mawell's equations [4, 5, 6, 7].

The key principle is that any energy gained by an electron from a system of electromagnetic fields must result from a corresponding decrease in the electromagnetic field energy.

Furthermore, the decrease in field energy is due to interference between the drive fields of the laser or electromagnetic cavity and the fields of the electron.

We are particularly interested in cases where the electron exhibits an energy gain after it has ceased to interact with the drive fields. In general, the total electromagnetic field energy can have been reduced only if a component of the radiation fields of the electron continues to occupy the same volume as the drive fields (until possible absorption of these fields by matter). The radiation fields include both the response fields induced by the motion of the electron in the drive fields, as well as the "spontaneous" radiation due to the electromagnetic interaction with the matter of the accelerating structure independent of the strength of the drive fields.

The interference of the spontaneous radiation with the drive fields is therefore responsible for that part of the acceleration which is linear in the strength of the drive fields. The interference between the drive field and the response radiation is quadratic in the strength of the drive field, and is usually neglected. However, in the case of vacuum laser acceleration there is no spontaneous radiation, and the leading acceleration term is quadratic in the strength of the laser field. The absence of vacuum laser acceleration that is linear in the laser field strength has been called the Lawson-Woodward theorem [8, 9].

The use of an energy argument is convenient when discussing such processes as vacuum laser acceleration, inverse Čerenkov acceleration, inverse transition radiation acceleration, *etc.*, for which a simple identification of the accelerating force is elusive (due in part to the lack of an analytic solution to Maxwell's equations for a focused laser beam).

Before exploring the case of vacuum laser acceleration in sec. 5, we first comment on particle acceleration in a static electric field (sec. 2), in a resonant cavity (3ec. 3) and in a plane electromagnetic wave (sec. 4).

# 2 Acceleration in a Static Field

The usual picture of acceleration in a static electric field of strength $E_0\hat{\mathbf{z}}$ in the $z$ direction is that the Lorentz force $\mathbf{F} = eE_0\hat{\mathbf{z}}$ acting on a charge $e$ over distance $L$ imparts energy $\Delta U_e = eE_0L$ to the charge.

Let us re-examine this case from a Maxwellian view that emphasizes field energy. When the charge is at distance $d$ from one of the electrodes that supports the field $E_0$, the electric field $\mathbf{E_e}$ of the charge can be obtained via the image-charge method. A short calculation in cylindrical coordinates shows that the cross term (interference term) of the field energy is (in Gaussian units),

$$\begin{aligned}
U_{\rm int} &= \int \frac{E_0\hat{\mathbf{z}}\cdot\mathbf{E}_e(\mathbf{r})}{4\pi}\, d{\rm Vol} \\
&= \frac{E_0\hat{\mathbf{z}}}{4\pi}\cdot\int\left(\frac{e\mathbf{r}_1}{r_1^3} - \frac{e\mathbf{r}_2}{r_2^3}\right)\, d{\rm Vol} \\
&= \frac{eE_0}{4\pi}\int_0^\infty dz\int_0^\infty \pi dr^2\left(\frac{z-L}{[r^2+(z-L)^2]^{3/2}} - \frac{z+L}{[r^2+(z+L)^2]^{3/2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{eE_0}{4} \int_0^\infty dz \left( \left\{ \begin{array}{ll} 2 & \text{if } z > L \\ -2 & \text{if } z < L \end{array} \right\} - 2 \right) \\
&= -eE_0 \int_0^L dz = -eE_0 L,
\end{aligned} \tag{1}
$$

with geometry as shown in Fig. 1. When the particle has traversed a potential difference $V = E_0 L$, it has gained energy $eV$ and the field has lost the same energy.[1]

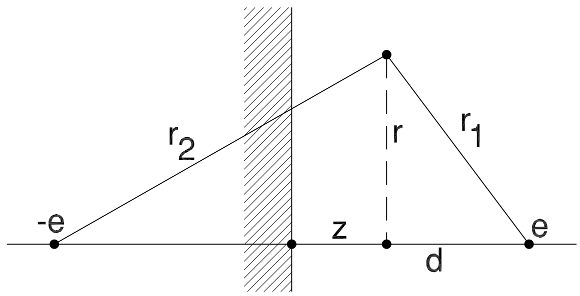


Figure 1: Electric charge $e$ and its image charge $-e$ at positions $(r, \theta, z) = (0, 0, \pm d)$ with respect to a conducting plane at $z = 0$. Vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed from the charges to the observation point $(r, 0, z)$.

In a practical "electrostatic" accelerator, the particle is freed from an electrode at potential $-V$ and emerges with energy $eV$ in a region of zero potential. However, the particle could not be moved to the negative electrode from a region of zero potential by purely electrostatic forces unless the particle lost energy $eV$ in the process, leading to zero overall energy change. An "electrostatic" accelerator must have an essential component (such as a battery) that provides a nonelectrostatic force that can absorb the energy extracted from the electrostatic field while moving the charge from potential zero so as to put the charge at rest at potential $-V$ prior to acceleration.

# 3 Acceleration in a Resonant Cavity

Consider a generic cavity with length $L$ along the direction of motion of an electron, taken to be the $z$ axis. The cavity is excited with a field of amplitude $E_0$ and wavelength $\lambda \gg L$. Consequently, $kL \ll 1$, where $k$ is the wave number. A practical accelerating cavity would have entrance and exit apertures of radius $a$ much smaller than the cavity length $L$, and hence much smaller than $\lambda$.

The usual view is that an electron gains energy $\Delta U_e \approx eE_0 L$ when it traverses the cavity with the appropriate phase. Let us see how this energy transfer is accounted for in a Maxwellian view.

An electron passing through an otherwise field-free cavity will radiate "spontaneously". In the case of a cavity containing a drive field as well, the interference between the drive

[1]See also [11].

field and the spontaneous radiation leads to energy transfers to the electron that are linear in the drive field. So, we concentrate our discussion on the "spontaneous" radiation in the cavity, and neglect the response ("stimulated") radiation.

First, consider the spontaneous radiation of an electron as is passes through the aperture in a metallic plate. In the limit of zero aperture radius, transition radiation arises, and the spectrum would extend up to the $\gamma\omega_P$, where $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentyz factor of the electron and $\omega_P$ is the plasma frequency of the metal. For an aperture of finite radius $a$, the radiation is simply that part of the transition radiation spectrum associated with radii greater than $a$.

Transition radiation can be thought of in terms of the Weizsäcker-Williams approximation [10]. As the electron suddenly emerges from the metal plate, its attraction to the image charge causes an acceleration, and hence radiation, perpendicular to the plate. The spectrum of the radiation is the Fourier transform of the pulse of electric field on the plate. At a radius $a$ from the trajectory of an electron with Lorentz factor $\gamma$, the radial electric field has strength $E \approx \gamma e/a^2$ and the pulse lasts for time $\approx a/\gamma c$, where $c$ is the speed of light. Hence, the spectrum extends up to wave number $k_{\rm max} \approx \gamma/a$, and the pulse energy is $U \approx E^2$ Vol $\approx \gamma^2 e^2/a^4 \cdot a^2 \cdot a/\gamma = \gamma e^2/a$. In effect, all of the pulse energy is radiated as transition radiation, and so the pulse spectrum is,

$$\frac{dU}{dk} \approx \frac{U}{k_{\rm max}} \approx e^2. \tag{2}$$

This result is independent of the radius $a$, and so applies to aperture radiation for all wave numbers up to $k \approx \gamma/a$ when the metal plate has an aperture of radius $a$. Since in the present case we seek interference between the aperture radiation and the cavity fields whose wavelength is much larger than $a$, eq. (2) holds for the relevant part of the spectrum.

The basic result (2) is usefully re-expressed in terms of photons of angular frequency $\omega = kc$. The number $n$ of photons radiated per energy interval is,

$$dn = \frac{dU}{\hbar\omega} \approx \frac{e^2}{\hbar c}\frac{d\hbar\omega}{\hbar\omega} = \alpha\frac{d\omega}{\omega}, \tag{3}$$

where $\hbar$ is Planck's constant and $\alpha = e^2/\hbar c$ is the fine-structure constant. Because this result follows from the Weizsäcker-Williams approximation, it will characterize a large variety of basic radiation process in addition to transition radiation. [We give examples later in the paper....]

The cavity has a second metallic plate at distance $L$ from the first. As the electron passes through the aperture of the second plate, it again emits aperture radiation, whose spectrum is identical to eq. (2), but whose maximum wave number is only $1/a$ for radiation in the backward direction. This backward radiation can be thought of as the result of reflection of part of the forward radiation from the second plate, and so has a 180° phase change compared to the forward radiation. Also, because the electron passes through the second aperture later than the first, there is a phase lag of,

$$\Delta\varphi = kL\left(\frac{1}{\beta} - \cos\theta\right) \approx kL\left(1 - \cos\theta + \frac{1}{2\gamma^2}\right), \tag{4}$$

between the two radiation fields, where $\theta$ is the relevant angle of emission of the radiation, and the approximation holds for relativistic electrons. To have the electric field vector parallel to the $z$ axis, we are interested in radiation for $\theta \approx 90°$. Then, $\Delta\varphi \approx kL \ll 1$, and the radiated electric field $\mathbf{E}_{1+2}$ from apertures 1 and 2 can be written as,

$$\mathbf{E}_{1+2} = \mathbf{E}_1(1 - e^{ikL}) \approx -ikLE_1\ \hat{\mathbf{z}}. \tag{5}$$

The spectrum of the spontaneous radiation into the cavity is then,

$$dU_{1+2} \approx k^2L^2\ dU_1 = e^2L^2k^2\ dk. \tag{6}$$

This energy excites the varies modes of the cavity, whose mode number density obeys,

$$dN \approx \text{Vol}\ k^2\ dk, \tag{7}$$

according to the Rayleigh-Jeans law. Hence, the energy $U_{\text{rad}}$ and electric field $\mathbf{E}_{\text{rad}}$ radiated into a single mode are related by,

$$U_{\text{rad}} \approx \frac{dU_{1+2}}{dN} \approx \frac{e^2L^2}{\text{Vol}} \approx \mathbf{E}_{\text{rad}}^2\text{Vol}, \tag{8}$$

and so,

$$\mathbf{E}_{\text{rad}} \approx -i\frac{eL}{\text{Vol}}\ \hat{\mathbf{z}}. \tag{9}$$

Finally, we consider the interference term in the total field energy when the cavity is driven by a wave of strength $\mathbf{E}_0$ in a particular mode, and find,

$$U_{\text{int}} \approx (\mathbf{E}_0 \cdot \mathbf{E}_{\text{rad}})\text{Vol} \approx -eE_{0,z}L, \tag{10}$$

for a suitable choice of phase of the drive field. Hence, as expected, the electron can extract energy $\Delta U_e \approx eE_{0,z}L$ from the cavity.

While the form of eq. (9) could have been anticipated on dimensional grounds, the more detailed argument shows the basic connection between particle acceleration in cavities and transition radiation. Since transition radiation arises when time dependent image charges are present, the role of image charges add to the conceptual link between acceleration in cavity fields and in static fields.

# 4 Acceleration by a Plane Wave

A plane wave can be approximated by the far zone of a spherical wave. The electric field amplitude rises slowly to strength $E_0$, then remains essentially constant for a long time, before slowly returning to zero. Can such a wave transfer net energy to an electron?

The answer is negative, as perhaps first noted by di Francia [12] and by Kibble [13]. A plane wave can, however, impart significant "temporary" acceleration to an electron when the latter is overtaken by the wave [14, 15, 16], but in general the electron loses its temporary energy gain on exiting the plane wave.

Because the motion of an electron in a plane wave is well known, the relation between the energy and momentum of the electron and of the field can be demonstrated in detail [17]. The oscillatory transverse momentum of the electron inside the plane wave is compensated by the interference between the electric field of the wave and the Coulomb field of the electron. The longitudinal momentum change and the energy gain of the electron while inside the wave are compensated by the interference between the wave fields and the oscillating fields of the electron. However, the relevant interference terms of the field energy and momentum densities are significant primarily within one wavelength of the electron, rather than in the far zone. Once the wave has passed the electron by, these near-zone interference terms vanish, along with the "temporary" acceleration of the electron.

It is convenient to use the example of a plane wave to introduce an invariant measure of whether the wave field is "strong" or "weak". We define the dimensionless Lorentz invariant $\eta$ by,

$$\eta = \frac{e\sqrt{\langle A_\mu A^\mu \rangle}}{mc^2} = \frac{eE_{0,\mathrm{rms}}}{m\omega c} = \frac{eE_{0,\mathrm{rms}}\lambda\!\!\!^{-}}{mc^2}, \tag{11}$$

where $A$ is the four-vector potential, the average is with respect to time, $m$ is the mass of the electron and $\omega = kc$ is the angular frequency of the wave. For weak fields, $\eta \ll 1$.

When a slowly modulated plane wave overtakes an electron, the latter oscillates transversely to the wave vector $\mathbf{k}$. The case of circular laser polarization is simpler. In its longitudinal (or average) rest frame the electron moves in a circle in the transverse plane characterized by Lorentz factors,

$$\gamma_\perp^\star = \sqrt{1+\eta^2}, \qquad \beta_\perp^\star = \frac{\eta}{\sqrt{1+\eta^2}}, \tag{12}$$

where quantities measured in this frame are denoted with the superscript $\star$. Thus, the transverse motion is relativistic for large $\eta$. For linear polarization, the motion is a "figure 8" [14] where $\gamma_\perp^\star$ and $\beta_\perp^\star$ vary over a cycle with average values given by (12).

An electron that is overtaken by a slowly modulated plane wave also takes on a "temporary" longitudinal velocity [13, 15, 16] described by Lorentz factors,

$$\gamma_\parallel(\gamma_\parallel\beta_\parallel) = \frac{\gamma_0\{1(\beta_0) + \eta^2[1+\beta_0]/2\}}{\sqrt{1+\eta^2}}, \tag{13}$$

where $\beta_0 = v_{0,z}/c$ and $\gamma_0 = 1/\sqrt{1-\beta_0^2}$, which apply to the electron's motion prior to the arrival of the wave, assumed to be parallel to the axis of the beam. To a reasonable approximation,

$$\gamma_\parallel \approx \gamma_0\sqrt{1+\eta^2}, \tag{14}$$

which for strong fields becomes $\gamma_\parallel \approx \gamma_0\eta$.

The "drift velocity" $\beta_\parallel$ of the electron inside the laser beam also follows from eqs. (13),

$$\beta_\parallel = \frac{\beta_0 + \eta^2(1+\beta_0)/2}{1+\eta^2(1+\beta_0)/2}. \tag{15}$$

For an electron initially at rest, $\beta_\parallel = \eta^2/(2+\eta^2)$, so that if, in addition, the field is weak, $\beta_\parallel \approx \eta^2/2$.

The (invariant) amplitude $r_\perp$ of the transverse oscillation of an electron inside the wave is,

$$r_\perp = \frac{\eta \lambda\!\!\!^{-\star}}{\sqrt{1+\eta^2}} \approx \frac{2\eta\gamma_\parallel \lambda\!\!\!^{-}}{\sqrt{1+\eta^2}} \approx \gamma_0 \eta \lambda\!\!\!^{-}. \tag{16}$$

A very useful concept that emerges from considerations of plane waves is the effective mass of the electron [13],

$$\overline{m} = \gamma_\perp^\star m = m\sqrt{1+\eta^2}. \tag{17}$$

When one wishes to average over the oscillatory transverse motion of an electron in the wave, this can be conveniently done by absorbing the transverse energy into the effective mass, with the stated result. The effective energy,

$$U_{\text{eff}} = \overline{m}c^2 = mc^2\sqrt{1+\eta^2}, \tag{18}$$

is then the total energy of the electron in a frame where the electron has zero longitudinal velocity (inside the wave).

For situations in which the electromagnetic wave is well approximated by a plane wave with with a slowly modulated amplitude, the effective energy (18) can be used to characterize the average behavior over a wave period of an electron as it enters and leaves the wave. For weak fields ($\eta \ll 1$), the effective energy is, to within a constant,

$$U_{\text{ponderomotive}} \approx \frac{mc^2\eta^2}{2}, \tag{19}$$

and is called the ponderomotive potential [13, 18]. Associated with this is an effective force,

$$\mathbf{F}_{\text{eff}} = -\boldsymbol{\nabla} U_{\text{ponderomotive}} = -\frac{mc^2}{2}\boldsymbol{\nabla}\eta^2, \tag{20}$$

which indicates that an electron is repelled from the high-field region of the wave.

# 5 Vacuum Laser Acceleration

By vacuum laser acceleration we mean energy gain of an electron that interacted with a laser beam in vacuum, and where any mirrors or lenses that define the laser beam are at distances large compared to the Rayleigh range from the focus.

## 5.1 Properties of Focused Laser Beams

We consider a configuration suitable for vacuum laser acceleration such as that illustrated in Fig. 2.

An axially symmetric laser beam of central wavelength $\lambda$ is focused to a waist of radius $w_0$. Asymptotically, the beam occupies a cone of half angle $\theta_0$ given by,

$$\theta_0 = \frac{\lambda}{\pi w_0} = \frac{2}{k w_0} \tag{21}$$

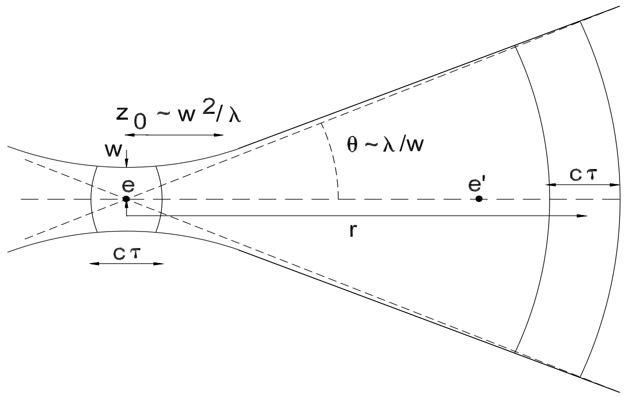


Figure 2: Scheme for vacuum laser acceleration of an electron by a laser pulse of length $\tau$ that is focused from the left to a waist of radius $w_0$ and has a Rayleigh range $z_0 = \pi w_0^2/\lambda$. An electron initially near the laser focus (left side of figure) is accelerated to the right and eventually falls behind the laser pulse in the far-field region (right side of figure).

which we also call the diffraction angle. The depth of focus is characterized by the Rayleigh range,

$$z_0 = \frac{w_0}{\theta_0} = \frac{\pi w_0^2}{\lambda} = \frac{2}{k\theta_0^2}. \tag{22}$$

The characteristic values $w_0$ and $\theta_0$ are twice the standard deviations of the corresponding Gaussian distributions of the laser intensity near the focus, and in the far field, respectively.

In this section and in secs. 5.3-5 we consider the fundamental Gaussian mode of a linearly or circularly polarized laser beam. In our initial discussion, the laser pulse is "short", meaning that its pulse length $\tau$ obeys $c\tau < z_0$, where $c$ is the speed of light.

The laser pulse overtakes an electron of negligible initial velocity in the vicinity of the laser focus. Much later, the laser pulse has advanced well beyond the electron and the Coulomb fields of the electron no longer interfere with those of the laser pulse. This situation is indicated schematically on the right side of Fig. 2, in which the center of the laser pulse is at distance $r \gg z_0$ from the focal point.

We denote the peak electric field of the laser pulse at the focus by $E_0$, and the electric field on the optic axis at distance $\mathbf{r}$ from the focus by $E_{\rm laser}(r)$. At a point $\mathbf{r} = (r, \theta, \phi)$, the far field of the laser is,

$$E_{\rm laser}(\mathbf{r}, t) = E_{\rm laser}(r) e^{-\theta^2/\theta_0^2} g(\varphi) e^{i\varphi}, \tag{23}$$

where the phase $\varphi$ is $kr - \omega t$ and $g$ is the pulse envelope function of characteristic width $\varphi_0 = c\tau/\lambda\!\!\!^-$, with $g(0) = 1$ and $g(\pm\infty) = 0$. A useful approximation to the laser pulse shape is,

$$g(\varphi) = \mathrm{sech}(\varphi/\varphi_0). \tag{24}$$

The energy $U_{\rm laser}$ in the laser pulse is the same at the focus and at distance $r$, and proportional to the square of the field strength times the pulse volume in the simplest approximation,

$$U_{\rm laser} \approx E_0^2 w_0^2 c\tau \approx E_0^2 \theta_0^2 z_0^2 c\tau \approx E_{\rm laser}^2(r) r^2 \theta_0^2 c\tau. \tag{25}$$

We suppress numerical factors such as $1/4\pi$ throughout this paper. The first forms of (25) can also be rewritten in terms of the dimenionless parameter $\eta$ defined in (11) as

$$U_{\rm laser} \approx \eta^2 mc^2 \frac{z_0}{r_e}\frac{c\tau}{\lambda}, \tag{26}$$

where $r_e = e^2/mc^2$ is the classical electron radius.

From (25) we also conclude that the electric field strength of the laser far from the focus is related by,

$$E_{\rm laser}(r) \approx \frac{z_0}{r} E_0 e^{i\Delta\varphi}, \tag{27}$$

for $\theta \lesssim \theta_0$, where the phase shift $\Delta\varphi$ between the far laser field and the field at the focus is not yet determined.

#### 5.1.1 The Gouy Phase Shift

The phase shift can be found by an inverse application of Kirchhoff diffraction theory, in which a field component $E_i(\mathbf{x})$ near the laser focus is reconstructed from the far field $E_i(\mathbf{r})$ [19],

$$E_i(\mathbf{x}) \approx \frac{ik}{2\pi}\int_S \frac{e^{-iks}}{s} E_i(\mathbf{r})\; d\text{Area}, \tag{28}$$

where $s$ is the magnitude of the vector $\mathbf{r} - \mathbf{x}$.

For example, a laser beam that is linearly polarized along the $x$ axis has far field $E_x(\mathbf{r})$ given by eq. (23), and a longitudinal component $E_z(\mathbf{r}) \approx -\theta\cos\phi E_x(\mathbf{r})$ so that the field $\mathbf{E}$ is transverse to the radius vector $\mathbf{r}$. Then, eq. (28) leads to the forms,

$$E_x = E_0 f g e^{-f\varrho^2} e^{i\varphi}, \qquad E_z = -i\theta_0 f \xi E_x, \tag{29}$$

where $E_0$ is the peak electric field at the focus, and,

$$f = \frac{1}{1+i\zeta} = \frac{1-i\zeta}{1+\zeta^2} = \frac{e^{-i\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}}, \tag{30}$$

in terms of the normalized spatial coordinates $\xi = x/w_0$, $\upsilon = y/w_0$, $\varrho^2 = \xi^2 + \upsilon^2 = \rho^2/w_0^2$ and $\zeta = z/z_0$. The fields given by eq. (29) satisfy Maxwell's equations with the neglect of terms of order $\theta_0^2$ [20], and supposing that $g'/g \ll 1$, *i.e.*, that the pulse width is large compared to a single cycle.

The phase, $-\tan^{-1} z/z_0$, of function $f$ is known as the Gouy phase [21], and implies a $90^\circ$ phase shift between the far field of the laser beam where $f \approx -iz_0/r$ and the center of the focus, half of which shift occurs within one Rayleigh range of the focus. Thus, eq. (27) can now be written as,

$$E_{\rm laser}(\mathbf{r}) \approx -i\frac{z_0}{r}E_0. \tag{31}$$

### 5.1.2 Phase and Group Velocity

The total phase $\varphi_{\rm tot}$ of the wave is,

$$\begin{aligned} \varphi_{\rm tot} &= kz - \tan^{-1}\frac{z}{z_0} + \frac{\rho^2 z z_0}{w_0^2(z^2+z_0^2)} - \omega t \\ &\approx kz - \frac{z}{z_0} + \frac{\rho^2 z}{w_0^2 z_0} - \omega t, \end{aligned} \tag{32}$$

where $\rho^2 = x^2 + y^2$ and the approximation holds for $\rho < w_0$ and $|z| < z_0$. Near the focus we can identify an effective wave number $k_{\rm eff}$ given by,

$$k_{\rm eff} \approx k - \frac{1}{z_0} + \frac{\rho^2}{w_0^2 z_0} = \frac{\omega}{c}\left[1 - \frac{\theta_0^2}{2}\left(1 - \frac{\rho^2}{w_0^2}\right)\right], \tag{33}$$

noting that $1/z_0 = k\theta_0^2/2$. Hence, the phase velocity, $v_p = \omega/k_{\rm eff}$, exceeds the speed of light whenever the transverse position $\rho$ of the electron is less than the waist $w_0$. This is not ideal for energy transfers between wave and electron. However, when discussing energy, it is more relevant to consider the group velocity of the laser beam, $v_g = d\omega/dk_{\rm eff}$.

In eq. (33), the optical parameters $\theta_0$ and $w_0$ depend on the frequency, in general. If the optical transport system fixes the diffraction angle $\theta_0$, then the waist varies with frequency according to $w_0 = 2c/\omega\theta_0$ and the group velocity is $v_g = c/[1 - \frac{\theta_0^2}{2}(1 - 3\rho^2/w_0^2)]$. If instead the waist is fixed, then,

$$v_g = \frac{c}{1 + \frac{\theta_0^2}{2}\left(1 - \frac{\rho^2}{w_0^2}\right)} \approx c\left[1 - \frac{\theta_0^2}{2}\left(1 - \frac{\rho^2}{w_0^2}\right)\right]. \tag{34}$$

In the latter case, the group velocity is less than $c$ near the focus, and seems to be the physically reasonable case.

Part of the complexity of $v_{\rm phase}$ and $v_{\rm group}$ is due to the fact that energy and momentum flow transversely as well as longitudinally in a focused laser beam. If we also define a velocity $v_{\rm flow}$ of the energy flow only along, the $z$ direction, we can obtain this by averaging $v_{\rm group}$ over $\rho$. Assuming expression (34) $v_{\rm group}$ is valid for $\rho \le w_0$, we find,

$$v_{\rm flow} = \frac{c}{1 + \theta_0^2/4} \approx c\left(1 - \frac{\theta_0^2}{4}\right). \tag{35}$$

The same result is found far from the focus by averaging $v_z = c\cos\theta$. See also Ref. [22].

### 5.1.3 The Formation Length

In our Maxwellian perspective we will be concerned with the electromagnetic radiation of an electron in a laser focus. A useful concept is the formation length, the distance over which the radiation moves one wavelength ahead of the electron, after which the radiation can be considered to be separate from the electron.

An electron inside a wave has longitudinal velocity $\beta_\|$ that differs from its initial longitudinal velocity $\beta_0$ according to eq. (15). Then, the formation length is given by,

$$L_{\rm formation} = \frac{\lambda}{1-\beta_\|} \approx \frac{\gamma_\|^2 \lambda}{2} \approx \gamma_0^2(1+\eta^2)\lambda\!\!\!^{-}. \tag{36}$$

For a strong laser beam with $\eta \gg 1$, the formation length is,

$$L_{\rm formation} \approx \eta^2 \gamma_0^2 \lambda\!\!\!^{-}. \tag{37}$$

## 5.2 General Remarks on Laser Acceleration

### 5.2.1 A Constraint on $z_0$

We first note a constraint that arises when it is desired that an electron interact with a laser beam near its focus. According to eq. (16), the amplitude $r_\perp \approx \gamma_0 \eta \lambda\!\!\!^{-}$ of the transverse oscillation of an electron of initial Lorentz factor $\gamma_0$ in a wave could be larger than the waist $w_0$ of the laser beam. In that case, the electron would exit the side of the laser pulse and be effectively lost from the accelerated electron beam. To avoid this, the laser waist should satisfy $w_0 \gtrsim \gamma_0 \eta \lambda\!\!\!^{-}$, and hence the Rayleigh range should obey $z_0 \gtrsim \gamma_0^2 \eta^2 \lambda\!\!\!^{-}$ Since $z_0 \gtrsim \lambda\!\!\!^{-}$ for any pratical focusing optics, this provides little constraint on weak laser fields. But for strong fields with $\eta \gg 1$, this constraint can be usefully written $z_0 \gtrsim L_{\rm formation}$, according to eq. (37). If the Rayleigh range is less than a formation length, the interaction between a strong wave and electron, and hence any laser acceleration, would be reduced by $z_0/L_{\rm formation}$.

On the other hand, if the Rayleigh range is longer than a formation length, the electron slips past the crest of the electromagnetic wave, and the acceleration would, in general, be less. Hence, we conclude that,

$$z_0 \approx L_{\rm formation} \tag{38}$$

is desirable for laser acceleration in strong fields.

### 5.2.2 An Estimate of Maximal Laser Acceleration

The distance over which a focused laser pulse is intense is roughly the Rayleigh range $z_0$. For a peak electric field $E_0$, the maximum energy that could be transferred to a charge $e$ over this length is $\Delta U_{\rm max} \approx eE_0 z_0$. However, laser pulses have largely transverse electric fields, so this maximum is not readily achieved. The peak component of the electric field along the optical axis is of order $\theta_0 E_0 = w_0 E_0/z_0$. Therefore, a better estimate is,

$$\Delta U_{\rm max} \approx eE_0\theta_0 z_0 = eE_0 w_0 = \frac{w_0}{\lambda\!\!\!^{-}} \eta mc^2. \tag{39}$$

We add to this the constraint (38) that the Rayleigh range be the formation length for best acceleration. For a strong field, this implies that $w_0 \approx \gamma_0 \eta \lambda\!\!\!^{-}$, so we obtain,

$$\Delta U_{\rm max} \approx \gamma_0 \eta^2 mc^2. \tag{40}$$

We will find below in secs. 5.5-6 that energy transfers of this order can be achieved with an "axicon" focus laser beam, provided the mirrors that create this focus are located approximately at $\pm z_0$ about the focal point. This form of laser acceleration is, therefore, not what we have called vacuum laser acceleration, but (as we will see) depends on the interference of the laser beam with the transition radiation from the electron.

Can the energy gain (40) be achieved in a vacuum laser acceleration configuration, with all optical media far from the laser focus? An optimistic answer is suggested by an argument analagous to Budker's view of acceleration in a pulsed helical delay line (can we give a reference?), and confirmed by the Maxwellian argument in sec. 5.4.

### 5.2.3 The Snowplow Model

Near the focus, the energy in the laser pulse flows with a velocity slightly less than the speed of light. If this pulse of energy overtakes an electron, the electron can be pushed forward by the ponderomotive force (20) and experience a net energy gain. Consider this in the rest frame of the energy flow of the laser pulse, in which the electron initially approaches the pulse. If the energy of the electron in this frame is less than the ponderomotive potential of the pulse, the electron is reflected elastically, and moves away from the pulse. Back in the lab frame the electron ends up with a large energy.

This argument does not apply to plane waves, whose energy flow velocity is the speed of light. In this case, the plane wave eventually passes over any electron in its way, and the trailing edge of the wave decelerates the electron back to its initial velocity.

We saw in eqs. (34)-(35) that the group velocity near the focus of the fundamental mode of laser beam is a function of the radius, with an average value of,

$$\beta_g \approx 1 - \frac{\theta_0^2}{4}. \tag{41}$$

The corresponding Lorentz factor is,

$$\gamma_g^2 \approx \frac{2}{\theta_0^2} = kz_0. \tag{42}$$

According to this view, there is a possibility of accelerating an electron provided the group velocity of the laser focus exceeds the longitudinal velocity of the electron. Equivalently, we require that,

$$\gamma_g > \gamma_0, \tag{43}$$

where $\gamma_0$ is the initial Lorentz factor of the electron, which is assumed to have initial velocity along the laser beam.

For a relativistic electron that satisfies condition (43), its energy $U^\star$ in the rest frame of the laser group velocity is $U^\star = \gamma^\star mc^2 = \gamma_g\gamma_0(1-\beta_g\beta_0)mc^2 \approx \gamma_g\gamma_0(1-\beta_0)mc^2 \approx \gamma_g mc^2/\gamma_0$. If the electron were initially at rest, then $\gamma_0 = 1$ and its energy in this frame would be $U^\star = \gamma_g mc^2 = \gamma_g mc^2/\gamma_0$.

If the electron is to be reflected from the wave, its longitudinal velocity must be reduced to zero before it penetrates to the center of the focus. Thus, its energy $U^\star$ must be less than the ponderomotive potential, *i.e.*, the effective energy (18) of transverse motion of an electron with zero longitudinal velocity at the laser focus. Hence, we need,

$$\sqrt{1+\eta^2} \gtrsim \frac{\gamma_g}{\gamma_0} > 1. \tag{44}$$

This cannot be satisfied for a weak field with $\eta \ll 1$. We expect significant vacuum laser acceleration only for strong laser fields that satisfy,

$$\eta^2 \gtrsim \frac{\gamma_g^2}{\gamma_0^2} \approx \frac{kz_0}{\gamma_0^2}, \tag{45}$$

or equivalently, $z_0 \lesssim L_{\mathrm{formation}}$.

We saw in sec. 5.2.1 that the amplitude of the transverse oscillation of the electron will be larger than the laser waist, and consequently ejected sideways from the laser focus, unless $z_0 \gtrsim L_{\mathrm{formation}}$. Comparing with the above, we again conclude that we should have $z_0 \approx L_{\mathrm{formation}}$ for best laser acceleration, which is equivalent to the requirement that,

$$kz_0 \approx \gamma_0^2 \eta^2. \tag{46}$$

In the group-velocity frame, the velocity of the electron is reversed by the wave, but the Lorentz factor $\gamma^\star$ remains the same. Back in the laboratory frame, the final Lorentz factor of the electron is,

$$\gamma_f \approx 2\gamma_g \gamma^\star \approx \frac{\gamma_g^2}{\gamma_0} \approx \frac{kz_0}{\gamma_0} \approx \eta^2 \gamma_0. \tag{47}$$

The energy gain of the electron is,

$$\Delta U_e = (\gamma_f - \gamma_0) mc^2 \approx \gamma_0 \eta^2 mc^2 \tag{48}$$

This saturates the estimate (40), and suggests that a configuration for vacuum laser acceleration exists.

### 5.2.4 Laser Energetics

The energy gain formula (40) is noteworthy in that the gain is a multiplicative factor, rather than additive as is the case for acceleration in an rf cavity of fixed length. However, to achieve the gain (40), the Rayleigh range and formation length must increase with the input electron energy according to eq. (37). Therefore, the energy gradient,

$$\frac{\Delta U_{\max}}{z_0} \approx \frac{mc^2}{\gamma_0 \lambda\!\!\!^{-}}, \tag{49}$$

falls off with increasing electron energy, and the process of laser acceleration will be less efficient at higher energies.

We elaborate on this issue by considering the required energy in the laser pulse.

For a laser focus with $z_0 \approx L_{\mathrm{formation}}$, the main part of the acceleration occurs while only a single cycle of the laser beam overtakes the electron. The volume of the laser pulse near its focus could then be as little as $w_0^2 \lambda\!\!\!^{-} \approx z_0 \lambda\!\!\!^{-2}$, with minimum pulse energy,

$$\begin{aligned} U_{\mathrm{laser,min}} &\approx E_0^2 z_0 \lambda\!\!\!^{-2} = \frac{z_0}{r_e} \eta^2 mc^2 \approx \gamma_0^2 \eta^4 \frac{\lambda\!\!\!^{-}}{r_e} mc^2 \\ &\approx \gamma_0^2 \eta^4 10^{-5}\ \mathrm{J}, \end{aligned} \tag{50}$$

where the numerical value is for an optical laser. Then, eq. (40) can also be written as,

$$\Delta U_{\max} \approx \frac{\gamma_0 r_e}{z_0} U_{\text{laser,min}} \approx \frac{r_e}{\gamma_0 \eta^2 \lambda\!\!\!^-} U_{\text{laser,min}}, \tag{51}$$

which also shows how the efficiency of acceleration decreases at higher electron energies.

As a numerical example, consider a laser focus with $\eta = 3$. Then, according to eq. (50),

$$U_{\text{laser,min}} \approx \gamma_0^2 10^{-3} \text{ J}, \tag{52}$$

Such a laser could increase the electron energy by a factor of 10 at a single focal point. The pulse energy needed to accelerate an electron from rest to $\gamma = 10$ is only 1 mJ, while the jump from $\gamma = 10$ to 100 would require 100 mJ, and the jump from $\gamma = 100$ to 1000 would require 10 J, *etc.*

The above estimates are based on the assumption of essentially single-cyle laser pulses. If it is more practical to deliver multicycle laser pulses, the pulses energies to achieve the maximal gain (40) are correspondingly larger. It will require a very large laser system to accelerate an electron to energies beyond 1 GeV.

#### 5.2.5 Electron Beam Stability

For laser acceleration to be the basis of a practical electron accelerator, it must accelerate a bunch of electrons in a manner that retains a usable bunch structure. This will require some care, as the process of laser acceleration tends to disperse the bunch both longitudinally and transversely.

The gain formala (40) indicates that an energy spread $\Delta\gamma_0$ of the input electron bunch is magnified to $\eta^2 \Delta\gamma_0$ during laser acceleration. This contrasts with acceleration in an rf cavity, where the energy spread in unchanged. The relative energy spread, $\Delta\gamma/\gamma$, is unchanged by laser acceleration, while it is reduced by acceleration in an rf cavity.

During laser acceleration, electrons will experience transverse momentum kicks of order $\gamma_\perp^\star \beta_\perp^\star mc = \eta mc$ in the longitudinal rest frame of the electron, recalling sec. 4. Transverse momentum is a relativistic invariant, so this holds in the laboratory frame as well. Since $mc = 0.511$ MeV/$c$, this defocusing effect is relatively modest for moderate values of $\eta$, but will require compensation via a set of focusing magnets between stages of laser acceleration.

Focusing of the electron beam could also be provided by a solenoid magnet coaxial with the optic axis. In principle, the cyclotron frequency could be matched to the laser frequency, both measured in the average rest frame of the electrons. Then, interference between the spontaneous radation of the electron in the solenoid field and the radiation induced by the laser would lead to an additional acceleration, linear in the laser field strength. This concept is the basis of invserse-free-electron-laser acceleration.

#### 5.2.6 The Maxwellian View

In the Maxwellian view, the energy gain of an electron that interacts with a laser beam is the negative of the interaction energy between the laser beam and the induced radiation of the electron. We evaluate this far from the laser focus as,

$$\begin{aligned} U_{\rm int} &= \frac{1}{2\pi} Re \int \mathbf{E}^\star_{\rm laser}(\mathbf{r}) \cdot \mathbf{E}_{\rm rad}(\mathbf{r})\ d{\rm Vol} \\ &\approx Re \int_{{\rm fixed}\ r} d{\rm Area} \int \mathbf{E}^\star_{\rm laser}(\mathbf{r},t) \cdot \mathbf{E}_{\rm rad}(\mathbf{r},t)\ d(ct). \end{aligned} \tag{53}$$

The radiated electric field is given by the Liénard-Wiechert equation,

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{e}{c}\left[\frac{\hat{\mathbf{s}} \times \left\{(\hat{\mathbf{s}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{s}})^3 s}\right]_{\rm ret}, \tag{54}$$

where $\mathbf{s} = \mathbf{r} - \mathbf{x}$ is the vector from the source point $\mathbf{x}$ to the observation point $\mathbf{r}$, and the quantity in brackets is evaluated at the retarded time $t' = t - s/c$. The far field of the laser is given by eq. (23), where we now write $\mathbf{E}_{\rm laser}(\mathbf{r}) = \mathbf{E}_{\rm laser}(r)e^{-\theta^2/\theta_0^2}$.

We change the variable of integration in eq. (53) from $t$ to $t'$, noting that $dt = dt'[1 - \boldsymbol{\beta}(t')\cdot\hat{\mathbf{s}}]$, to find,

$$U_{\rm int} \approx \frac{e}{2\pi c} Re \int d{\rm Area}\ \frac{\mathbf{E}_{\rm laser}(\mathbf{r})}{s} \int g(\varphi)e^{-i\varphi} \frac{\hat{\mathbf{s}} \times [(\hat{\mathbf{s}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{s}})^2}\ d(ct'). \tag{55}$$

This can be integrated by parts using the relations,

$$\frac{\hat{\mathbf{s}} \times [(\hat{\mathbf{s}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{s}})^2} = \frac{d}{dt'}\left[\frac{\hat{\mathbf{s}} \times (\hat{\mathbf{s}} \times \boldsymbol{\beta})}{1 - \boldsymbol{\beta}\cdot\hat{\mathbf{s}}}\right] \tag{56}$$

and $d\varphi/dt' \approx -\omega(1-\boldsymbol{\beta}\cdot\hat{\mathbf{s}})$, giving,

$$U_{\rm int} \approx Re\ \frac{ik}{2\pi} \int d{\rm Area}\ \frac{e\mathbf{E}_{\rm laser}(\mathbf{r})}{s} \int [g(\varphi) + ig'(\varphi)]e^{-i\varphi}[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}]\ d(ct'). \tag{57}$$

The Gaussian beam described by eq. (29) satisfy Maxwell's equations only if $g'/g$ is negligible, so we ignore the term in $g'$ eq. (57).

To a good approximation, $\mathbf{E}_{\rm laser}\cdot\hat{\mathbf{s}} = 0$. Then, using the inverse diffraction integral (28), we find,

$$U_{\rm int} \approx -Re \int e\mathbf{E}_{\rm laser}(\mathbf{x}, t') \cdot \mathbf{v}\ dt', \tag{58}$$

where $\mathbf{v} = d\mathbf{x}/dt'$ is the velocity of the electron. Thus, the interference energy is indeed the negative of the energy gain of the electron in the laser field.

The Maxellian view gives us the option of evaluating the energy gain of the electron via the interference term (53) in the far field, as an alternative to direct evaluation of eq. (58) near the laser focus.

## 5.3 Acceleration by a Weak Laser

We begin our discussion of laser acceleration from the Maxwellian viewpoint with the case of a weak laser beam with $\eta \ll 1$ and pulse length $\tau$ such that $c\tau \lesssim z_0 \approx \lambdabar/\theta_0^2$. The electron is initially at rest. For simplicity, we consider an electron initially on the optic axis, with $|z| \lesssim z_0$

*According to the snowplow model of sec. 5.2.3, we expect no acceleration here, since the energy of the electron in the group-velocity frame is* $U^\star = \gamma_g mc^2 = \sqrt{2}mc^2/\theta_0$*, which is less than the the ponderomotive energy* $mc^2\sqrt{1+\eta^2} \approx mc^2(1+\eta^2/2)$ *so long as* $\theta_0 < \sqrt{2}/2$*. In this case, the laser pulse passes over the electron, and the latter is decelerated back to zero velocity by the trailing edge of the pulse.*

While the electron is inside the laser pulse, it radiates electromagnetic waves, and takes on longitudinal velocity $\beta_\| \approx \eta^2$ according to eq. (15). The longitudinal displacement is roughly $\Delta z \approx \eta^2 c\tau \lesssim \eta^2\lambdabar/\theta^2 < \lambdabar$. The amplitude of transverse oscillations is $\eta\lambdabar$ according to eq. (16), which is also less than $\lambdabar$ for a weak laser field.

Even when the laser pulse has passed the electron, the radiation fields of the electron still interfere with the laser pulse. For weak lasers fields, the radiation fields of the electron can be approximated as due to electric dipole radiation when the electron was at its initial position, since the electron moves less than a wavelength as the pulse passes by. The second time derivative of the electric dipole moment $\mathbf{d}$ of the electron is related by,

$$\ddot{\mathbf{d}}(\mathbf{x},t) = e\ddot{\mathbf{x}} \approx \frac{e^2}{m}\mathbf{E}(\mathbf{x},t) = c^2 r_e \mathbf{E}(\mathbf{x},t), \tag{59}$$

where $\mathbf{x}$ is the position of the electron. The radiation field $\mathbf{E}_{\rm rad}(\mathbf{r},t)$ of the electron at position $(r \gg z_0, \theta \sim \theta_0, \phi)$ in a spherical coordinate system is obtained from $\ddot{\mathbf{d}}$ when evaluated at the retarded time,

$$\begin{aligned} t' &= t - \frac{|\mathbf{r}-\mathbf{x}|}{c} \approx t - \frac{r - \hat{\mathbf{r}}\cdot\mathbf{x}}{c} \\ &= t - \frac{r}{c} + \frac{z\cos\theta}{c} \\ &\approx t + \frac{z}{c} - \frac{r}{c} - \frac{z\theta^2}{2c}. \end{aligned} \tag{60}$$

The total phase of $\mathbf{E}_{\rm rad}$ is then found from eq. (32) to be,

$$\varphi_{\rm tot} \approx kr - \omega t - \frac{z}{z_0} + \frac{kz\theta^2}{2} \tag{61}$$

In calculating the interference between the radiated field and the laser field, we average over angle $\theta$, weighting by the electric field of the laser beam in the far zone, which has the effect of replacing $\theta$ in (61) by $\theta_0$. Since $z_0 = 2/k\theta_0^2$, the averaged phase of the radiation is just $\langle\varphi_{\rm tot}\rangle = kr - \omega t$. A similar, but more intricate calculation yields the same result if the electron is initially off the optic axis.

Since the dipole moment is essentially orthogonal to the wave vector in the far-field region of the laser pulse, the far radiation field for $\theta \lesssim \theta_0$ is,

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{(\ddot{\mathbf{d}}(\mathbf{x},t')\times\hat{\mathbf{k}})\times\hat{\mathbf{k}}}{c^2 r} \approx -\frac{\ddot{\mathbf{d}}(\mathbf{x},t')}{c^2 r} = -\frac{r_e}{r}\mathbf{E}(\mathbf{x},t')$$

$$\approx \quad -\frac{r_e}{r}\frac{\mathbf{E}_0}{\sqrt{1+z^2/z_0^2}} = -\frac{e\eta}{r\lambda\!\!\!^{-}}\frac{\hat{\mathbf{E}}_0}{\sqrt{1+z^2/z_0^2}}\,, \tag{62}$$

where,

$$\varphi_{\mathrm{rad}}(\rho, z) \equiv \frac{z}{z_0}\left(1 - \frac{\rho^2}{w_0^2}\right), \tag{63}$$

where we have suppressed the phase $\varphi = kr - \omega t$.

This result implies that radiation emitted by an electron near the laser focus is 90° out of phase with the laser field in the far zone, eq. (31), since the laser field experiences the Gouy phase shift between the focus and far zone (associated with the focal spot having finite transverse extent), while the radiation from a point electron does not.

Therefore, there is no interference between the laser field and the radiation from the electron in the far zone. The electromagnetic field energy is constant (neglecting the small amount of energy in the radiation fields themselves), and so the mechanical energy of the electron must be constant (with the neglect of the small radiation reaction).

We have neglected the radiation fields caused by the longitudinal field, $E_z$, of eq. (29). The polarization of this radiation is radial, and so it does not contribute to interference with the far field of the laser beam after integration over azimuth, for any polarization of the laser in its fundamental mode. Hence, the longitudinal component of the fundamental laser mode cannot contribute to vacuum laser acceleration. We consider laser modes matched to the radiation caused by the longitudinal component of the laser field in secs. 5.5-7.

Thus, the Maxwellian view confirms the prediction of the snowplow model that there is no acceleration of an electron by a weak laser field.

## 5.4 Acceleration by a Strong Laser

In a strong laser field ($\eta \gtrsim 1$), the radiation of the electron is considerably modified. The transverse motion of the electron is relativistic, and the frequency spectrum of the radiation is dominated by higher harmonics. The case of circular laser polarization is simpler. In its longitudinal rest frame (boost = $\gamma_\parallel$), the electron moves in a circle in the transverse plane, and the radiation can then be called synchrotron radiation. In this frame, the transverse motion can be characterized by Lorentz factors (12), and so is relativistic for large $\eta$. For linear polarization, the motion is a "figure 8" [14] where $\gamma_\perp$ and $\beta_\perp$ vary over a cycle with average values given by (12).

We are only interested in the strength of the electric field at frequency $\omega$, as only this component can interfere with the far laser field. [Strictly, this statement implies that the interference extends over many cycles so that the interference between frequencies $\omega$ and $n\omega$ averages to zero for $n > 1$. But we will eventually see that in practice the interference region is less than one wavelength long, so higher harmonics are not totally negligible.]

From eq. (74.9) of Ref. [14] with harmonic number $n = 1$, we have that the total power radiated at frequency $\omega$ by an electron in circular motion is,

$$\frac{dU_\omega^\star}{dt^\star} \approx \frac{e^4 E_{\mathrm{laser}}^{\star 2}}{\gamma_\perp^{\star 2} m^2 c^3} = \frac{e^4 E_{\mathrm{laser}}^{\star 2}}{m^2 c^3 (1+\eta^2)}, \tag{64}$$

in the longitudinal rest frame of the electron, since the Bessel functions have values of order unity in this case. The radiated power is a Lorentz invariant, and the electric field transforms as $E^\star = \gamma_\parallel(1-\beta_\parallel)E \approx E/\gamma_\parallel$. Hence, in the lab frame we have

$$\frac{dU_{\omega,\mathrm{source}}}{dt} \approx \frac{e^4 E^2_{\mathrm{laser,\ source}}}{\gamma_\parallel^2 m^2 c^3(1+\eta^2)}, \tag{65}$$

for the radiation rate measured at the source. An observer in the far field near the optic axis detects a rate that is higher than (65) by a factor $1/(1-\beta_\parallel) \approx \gamma_\parallel^2$ because the electron is moving with speed $\beta_\parallel$ towards the observer. Thus,

$$\frac{dU_{\omega,\mathrm{far}}}{dt} \approx \frac{e^4 E^2_{\mathrm{laser,\ source}}}{m^2 c^3(1+\eta^2)} \approx \frac{e^2\eta^2 c}{\lambda\!\!\!^{-2}(1+\eta^2)}. \tag{66}$$

We can now estimate the strength of the radiated electric field in the far field by noting that the radiation pattern lies within a cone of half angle $\theta \approx 1/\gamma_\parallel$. The energy at frequency $\omega$ contained within a pulse of duration $\Delta t$ as observed in the far field is,

$$\begin{aligned}\Delta U_{\omega,\mathrm{rad}} &\approx E^2_{\omega,\mathrm{rad}}(r) r^2\theta^2 c\Delta t \approx \frac{E^2_{\omega,\mathrm{rad}}(r) r^2 c\Delta t}{\gamma_\parallel^2} \\ &\approx \frac{e^2\eta^2 c\Delta t}{\lambda\!\!\!^{-2}(1+\eta^2)},\end{aligned} \tag{67}$$

using eq. (66). Hence,

$$E_{\omega,\mathrm{rad}}(r) \approx -\frac{\gamma_\parallel e\eta e^{-i\Delta\varphi}}{r\lambda\!\!\!^{-}\sqrt{1+\eta^2}} \approx -\gamma_0 \frac{e\eta}{r\lambda\!\!\!^{-}} e^{-i\Delta\varphi}, \tag{68}$$

using (14), where the phase $\Delta\varphi$ between the radiated field and the laser field in the far zone is not yet determined. In the weak-field limit, eq. (68) agrees with eq. (62). This suggests, but does not prove, that $\Delta\varphi$ is zero...

*More work on the phase factor needed here....*

We can now calculate the interference energy between the radiation from the electron and the far field of the laser beam, similarly to eq. (57). For a strong wave, we saw that $\theta_0 \approx 1/\gamma_\parallel \approx 1/\gamma_0\eta$. Also, we have noted in sec. 5.2.1 that the Rayleigh range should be approximately the formation length. This has the consequence that the radiation from the electron has extent of only $c\tau \approx \lambda\!\!\!^{-}$. Then, we find,

$$U_{\mathrm{int}} \approx -\frac{\lambda\!\!\!^{-}}{\gamma_0^2\eta^2}\gamma_0 E_0^2 z_0 r_e \Delta\varphi = -\gamma_0\eta^2\Delta\varphi mc^2. \tag{69}$$

The energy gain of the electron is then,

$$\Delta U_e \approx \gamma_0\eta^2\Delta\varphi mc^2, \tag{70}$$

which is the maximal energy gain estimated in eqs. (40) and (48), to within the phase factor $\Delta\varphi$.

If we accept the conjecture that $\Delta\varphi$ is given by eq. (63), then we conclude that the maximal acceleration occurs for an electron that moves with the wave at distance $z_0$ ahead of the peak of the pulse.

*More work needed to clarify this.....*

*10/11/10 For snowplow acceleration, don't want electron to pentrate all the way to the center of the laser pulse. Suppose it only penetrates to $z \approx z_0$. The Gouy phase here is about 1 radian, which is not too far out of phase with the far fields. So, there can be considerable interference between radiation and the far laser field.*

*However, for strong acceleration, the radiation is so strong (????) that the electron doesn't actually gain much net energy.....*

*Must recall that radiation is like synchrotron radiation, $\gamma^2$, and lasts for time of order $z_0/c$.....*

A result that appears to be very similar to this has been reported by Wang *et al.*[7] on the basis of a numerical calculation.

## 5.5 Acceleration by an Axicon Laser Mode

In secs. 5.1-4 we assumed that the electron's trajectory lies along the optic axis of the laser beam. This implies that eventually the electron would pass through the lens or mirrors that shape the laser beams. This might be awkward in practice, especially for an intense beam of electrons. Therefore, it is natural to consider configurations in which the the optic axis of the laser makes an angle $\theta_1 > \theta_0$ to the electron beam. If a second laser beam is added at angle $-\theta_1$ and properly phased with respect to the first laser beam, the electric field on axis can be made purely longitudinal, which appears to be desirable [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

Such a configuration of laser beams is sometimes called an axicon-focus beam, the idealized version of which is a cylindrical Gaussian laser mode [9, 35, 36, 37]. In general, such modes exist with two polarizations of the transverse electric field, radial and azimuthal.

In notation like that of eqs. (29-30), the simplest radially polarized mode, the (0,0) mode, is (to first-order in the diffraction angle $\theta_0$),

$$E_\rho = \varrho F_0, \qquad E_\phi = 0, \qquad E_z = i\theta_0(1 - f^2\varrho^2)F_0, \tag{71}$$

where,

$$F_0 = E_0 \frac{e^{-\varrho^2/(1+\zeta^2)} g(\varphi)}{1+\zeta^2} e^{i(\varphi - 2\tan^{-1}\zeta)}, \tag{72}$$

$f = 1/(1 + i\zeta)$ and $\varphi = kz - \omega t$.

The (0,1) cylindrical mode with radial polarization is,

$$E_\rho = \varrho F_1, \qquad E_\phi = 0, \qquad E_z \approx i\theta_0 F_1, \tag{73}$$

where,

$$F_1 = \left(1 - \frac{\varrho^2}{1+\zeta^2}\right) e^{i\phi} F_0, \tag{74}$$

and $\phi$ is the azimuthal angle about the $z$ axis. The (0,1) azimuthally polarized cylindrical mode is,

$$E_\rho = 0, \qquad E_\phi = \varrho F_1, \qquad E_z = -\theta_0 F_1/2. \tag{75}$$

In all three modes listed the transverse fields vanish on the $z$ axis, and the Gouy phase shift between the focus and the far field is 180°. The longitudinal field at the focus is 90° out of phase with the far transverse field for the case of radial polarization, but is in phase for azimuthal polarization.

If such a laser beam encounters an electron lying on the $z$ axis, the latter will be accelerated by the field $E_z$, leading to radiation fields with radial polarization. Hence, there would be no interference between the radiation fields of the electron and the far fields of an azimuthally polarized laser beam. In turn, this implies that the acceleration is only temporary and does not result in any net transfer of energy to the electron.

The polarization of the radiation is matched to that of the radially polarized mode, but the radiation is aways exactly 90° out of phase with the far field of the laser beam. Again there can be no net energy transfer from the laser beam to the electron. This result can also be deduced from direct integration of the equation of motion of the electron over an infinite path length for arbitrary phasing of the electron compared to the laser beam (see eq. (76) [9, 31]). Another explanation for this null result is that since the phase velocity of the laser beam exceeds the speed of light, the wave always slips past the electron over a long enough path so as to cancel any temporary energy transfer.

These conclusions hold whether the electron is relativistic or nonrelativistic, and whether the laser field is weak or strong, in contrast the the more complex case of the fundamental laser mode. The conclusions are unchanged by the addition of multiple frequencies to the laser beam [33], and are consistent with the Lawson-Woodward theorem [8, 9]. Vacuum laser acceleration is impossible in the cylindrical laser modes, which appeared the best matched to the problem.

## 5.6 Acceleration in a Laser Cavity

In practice, a laser beam exists only over a finite length in $z$, typically defined by mirrors. Clearly, electrons could be accelerated in the resulting optical cavity, if the transition radiation from the cavity walls interferes with the laser beam. The acceleration would be linear in the laser field strength, as is the case for acceleration in an rf cavity. By our definition, this is not, however, vacuum laser acceleration (even if the cavity is under vacuum) since the cavity walls play an essential role.

A difference between a laser cavity and an rf cavity is that the cavity length is typically much longer than a laser wavelength, while usually shorter than an rf wavelength. Also, the phase velocity in an rf cavity can be adjusted to match that of a charge particle, while the phase velocity in a laser cavity will typically exceeds the speed of light. Hence, useful laser cavity acceleration will occur only over a distance such that the wave advances by about one reduced wavelength past the electron. This distance was introduced in sec. 5.1.3 as the formation length, $L_{\rm formation} \approx \gamma^2 \lambda\!\!\!^{-}$, where $\gamma$ is the Lorentz factor of the electron. Again, we conclude that optimal energy transfer between a laser beam and an electron occurs when the Rayleigh range is equal to the formation length. Accordingly, the optimal diffraction angle

$\theta_0$ is approximately $1/\gamma$.

A direct calculation of the energy gain from $E_z$ of a long ($g = 1$) radially polarized laser mode, eqs. (71-72), over path length $2L$ confirms this [9, 31, 37],

$$\begin{aligned} \Delta U_e(L) &= eE_{0,z} \int_{-L}^{L} dz \, \frac{z_0^2}{z^2 + z_0^2} \cos[2\tan^{-1}(z/z_0)] \\ &= 2eE_{0,z} z_0 \frac{L z_0}{L^2 + z_0^2}. \end{aligned} \tag{76}$$

The maximum energy gain is attained at the matching condition, $L = z_0$, with the same result as eqs. (39)-(40),

$$\Delta U_{e,\max} = eE_{0,z} z_0 = eE_0 \theta_0 z_0 = eE_0 w_0. \tag{77}$$

To achieve this maximum, the Rayleigh range should also be equal to the formation length, so as to avoid the problem of phase slippage (which was ignored in calculation (76)). Then $\theta_0 = 1/\gamma$ and the energy gradient is $eE_0/\gamma$, so that laser cavity acceleration becomes less and less effective at higher energies.

Let us apply the Maxwellian argument for cavity acceleration, now examining what happens if the separation $2L$ between the walls of the cavity becomes large compared to a wavelength. We look for interference between the transition radiation from the first cavity wall and the transversely polarized field of the laser near the focus.

The spectrum of transition radiation from the first wall is $dU_1 \approx e^2 dk = e^2 d\omega/c$, according to eq. (2). If the temporal pulse length of the laser is $\tau$ (which we assume to be less than $z_0/c$), then the corresponding frequency band is $d\omega \approx 1/\tau$. Hence, the total energy of the transition radiation within the bandwidth of the laser is $U_1 \approx e^2/(c\tau)$. Furthermore, the effective pulse length of the transition radiation emitted into this bandwidth is also $\tau$.

This radiation must also overlap the laser pulse in space. For an electron moving with Lorentz factor $\gamma$, the characteristic angle of the radiation is $\theta \approx 1/\gamma$. If the first wall is distance $L$ from the laser focus, the transition radiation has spread over radius $L\theta \approx L/\gamma$ when it reaches the focus. Hence, the volume of the transition radiation that could interfere with the laser is roughly $c\tau(L/\gamma)^2$. The corresponding field strength $E_1$ is related by,

$$U_1 \approx c\tau \frac{L^2}{\gamma^2} E_1^2 \approx \frac{e^2}{c\tau}, \tag{78}$$

so that,

$$E_1 \approx \frac{\gamma e}{c\tau L}, \tag{79}$$

to within a phase factor.

The transition radiation interferes with the laser field near the focus, whose field strength is called $E_0$. The length of the interaction volume is $c\tau$, while its radius is the minimum of the laser waist, $w_0$, and the radius $L/\gamma$ of the transition radiation pattern.

First, consider small cavity length $L$, in which case the overlap radius is $L/\gamma$. The interference term in the total field energy is,

$$U_{\rm int} \approx -E_0 E_1 c\tau \frac{L^2}{\gamma^2} \approx -eE_0 \frac{L}{\gamma}, \tag{80}$$

for a suitable choice of phase of the electron relative to the laser beam. If the focus of the laser beam is matched to the Lorentz factor of the electron, we have $\theta_0 = w_0/z_0 = 1/\gamma$ for the diffraction angle. The energy gain by the electron is then,

$$\Delta U_e = -U_{\rm int} \approx eE_0 w_0 \frac{L}{z_0} \approx eE_{0,z} L. \tag{81}$$

The energy gain increases with cavity length until it reaches the maximum stated in eq. (77) at $L = z_0$, beyond which the transition radiation no longer overlaps fully with the laser beam.

For laser cavity length $L > z_0$, the overlap radius is $w_0$. The interference term in the total field energy is then,

$$U_{\rm int} \approx -E_0 E_1 c\tau w_0^2 \approx -eE_0 \frac{\gamma w_0^2}{L}, \tag{82}$$

for a suitable choice of phase of the electron relative to the laser beam. Again, we suppose that the focus of the laser beam is matched to the Lorentz factor of the electron, so that $\gamma w_0 \approx z_0$. The energy gain by the electron is then,

$$\Delta U_e = -U_{\rm int} \approx eE_0 w_0 \frac{z_0}{L} \approx eE_{0,z} z_0 \frac{z_0}{L}. \tag{83}$$

The energy gain is suppressed for cavity lengths greater than the Rayleigh range.

The Maxwellian argument emphasizes the transverse, rather than the longitudinal, field of the laser, since it is the former that interferes strongly with the radially polarized transition radiation. Then, we conclude that the laser beam should be radially polarized as well; a plane polarized laser could cause no laser cavity acceleration. Finally, we connect acceleration with longitudinal electric field by noting that radially polarized laser modes have nonvanishing longitudinal electric fields on axis, in contrast to plane-polarized modes. We also conclude that azimuthally polarized laser beams will not accelerate electrons, even though they have nonvanishing longitudinal electric field on axis. [**Problem:** A direct calculation using $E_z$ from eq. (74) seems to give the same results as (76), if the phase of the electron is shifted by $90^\circ$, which we are free to do????]

The Maxwellian argument also allows us to appreciate an important point not readily discerned from direct integration of the equation of motion. A practical laser cavity accelerator will likely contain apertures in the end mirror to let the electron beam pass. Those apertures must not be so large that the transition radiation spectrum no longer covers the laser frequency, otherwise the interference term would disappear and the electron could not extract energy from the system. In other words, the aperture must not perturb the longitudinal field too greatly or the direct integration that led to eq. (76) would be inaccurate. Recall that the cutoff in the transition radiation spectrum is at wave number $k \approx \gamma/a$ in case of an aperture of radius $a$. Hence, we must have,

$$a \lesssim \gamma \lambda\!\!\!^{-} \tag{84}$$

for laser cavity acceleration to be effective. This is a rather serious restriction for electrons of energies up to a few GeV in an optical laser cavity accelerator.

# 6 Acceleration by Inverse Radiation

A laser beam can transfer energy to a copropagating electron whenever the electron emits "spontaneous" radiation that can interfere with the laser field. The energy gain will be linear in the electric field strength of the laser. Numerous schemes have been proposed for the needed "spontaneous" emission, some of which are reviewed in this section.

## 6.1 Inverse Čerenkov Acceleration

Among the proposed variants of laser cavity acceleration, many involve the use of a gas with index of refraction $n$, nominally to reduce the phase velocity of the laser beam to $c/n$ to match the velocity $c\beta < c$ of the electron [1, 9, 27, 28]. From the Maxwellian veiwpoint, the role of the gas is to induce the electron to emit (radially polarized) Čerenkov radiation. Then, if an axicon-focus laser beam is arranged with diffraction angle that overlaps the Čerenkov cone, interference and acceleration are possible.

Precise matching of the phase velocity is not required, if the path length is short enough that the phase slippage is length than $\lambda$. Indeed, the condition that $\beta = 1/n$ is the threshold for Čerenkov radiation, so the radiation, the interference and the energy transfer would all vanish in this case.

## 6.2 Cyclotron Resonance Acceleration

An electron in a uniform magnetic field moves in a helix about a field line, and emits a variant of synchrotron radiation. If a laser beam propagates along the field line and has the same frequency as the synchrotron radiation, interference can occur and the electron can be accelerated. This scheme has been discussed by several authors [38, 39, 40, 41].

## 6.3 Inverse Bremsstrahlung Acceleration

When an electron moves through a transverse, static electric or magnetic field it emits radiation that could be called a kind of bremstrahlung or (perhaps more properly) synchrotron radiation. If a laser beam propagates along with the electron and has a frequency that overlaps with the radiation spectrum induced by the static field, interference and acceleration can occur. This process is often called inverse bremsstrahlung acceleration [42, 43, 45, 46].

## 6.4 Inverse Free-Electron-Laser Acceleration

Lots of refs......

## 6.5 Inverse Smith-Purcell Radiation

[8, 47, 48, 49, 50, 51]

## 6.6 Plasma Beat-Wave Acceleration

The beat goes on.....

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