Princeton/$\mu\mu$/97-4
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July 9, 1997, revised July 18, 1997

Accuracy of Measurements in the Muon-Collider Cooling Experiment

1 Combined Accuracy

Ignoring correlations, the 6-dimensional emittance ϵ of the muon beam is the product of the variances (second-moments) of the projections of the muon population on the 6 axes x, x', y, y', E and t. In this note we label a variance by σ_i^2 where i indicates one of the six phase-space axes. We call σ_i the rms width. The uncertainty in σ_i is labelled δ_{σ_i}. Then the emittance is

$$\epsilon = \prod_{i=1}^{6} \sigma_i, \tag{1}$$

and the corresponding uncertainty is

$$\frac{\delta_\epsilon}{\epsilon} = \sqrt{\sum_{i=1}^{6} \left(\frac{\delta_{\sigma_i}}{\sigma_i} \right)^2}. \tag{2}$$

Supposing the relative uncertainty is the same in all six projections, we have

$$\frac{\delta_\epsilon}{\epsilon} = \sqrt{6} \frac{\delta_\sigma}{\sigma}. \tag{3}$$

In the cooling experiment it is proposed to demonstrate a factor of two reduction in the 6-d emittance. Before this large a reduction is observed we will likely pass through stages with smaller reduction. Hence the measurement uncertainty, δ_ϵ/ϵ, should be much less than 1/2; a goal of 3% uncertainty has been set. Then eq. (3) implies that the relative uncertainty in the rms width of the projection of phase space onto each of the six axes should be only 1%.

2 Effect of Detector Resolution

For several phase-space axes the desired relative uncertainty in the rms width of 1% is smaller than the ratio of the rms width σ_D of the detector resolution function to the rms width σ_i off the projected distribution, That is, $\sigma_D/\sigma_i > 0.01$. Typically, multiple scattering is the cause of poor detector resolution. How well can σ_i be measured in this case?

In the gaussian approximation, the observed rms width σ_O of the projected phase-space distribution is sum in quadrature of the 'true' rms width σ_i and the rms width σ_D of the detector resolution:

$$\sigma_O^2 = \sigma_i^2 + \sigma_D^2. \tag{4}$$

We suppose that σ_D is known to an accuracy δ_{σ_D}. Then we extract the desired rms width σ_i according to

$$\sigma_i^2 = \sigma_O^2 - \sigma_D^2. \tag{5}$$

We now wish to characterize the uncertainty δ_{σ_i}. From eq. (5) we find

$$\delta^2_{\sigma_i^2} = \delta^2_{\sigma_O^2} + \delta^2_{\sigma_D^2}. \tag{6}$$

Next, the uncertainty in the observed variance σ_O^2 after a set of N measurements is[1]

$$\delta_{\sigma_O^2} = \sqrt{\frac{2}{N}}\sigma_O^2 = \sqrt{\frac{2}{N}}\left(\sigma_i^2 + \sigma_D^2\right). \tag{7}$$

Then noting that $\delta_{\sigma^2} = 2\sigma\delta_\sigma$ we find the key result:

$$\left(\frac{\delta_{\sigma_i}}{\sigma_i}\right)^2 = \frac{1}{2N}\left(1 + \frac{\sigma_D^2}{\sigma_i^2}\right)^2 + \left(\frac{\sigma_D}{\sigma_i}\right)^4 \left(\frac{\delta_{\sigma_D}}{\sigma_D}\right)^2. \tag{8}$$

2.1 Perfectly Known Resolution

In the limit that the detector resolution is completely understood we have $\delta_{\sigma_D} = 0$ and the relative uncertainty in the rms width σ_i is

$$\frac{\delta_{\sigma_i}}{\sigma_i} = \sqrt{\frac{1}{2N}}\left(1 + \frac{\sigma_D^2}{\sigma_i^2}\right). \tag{9}$$

If the detector resolution σ_D is larger than the rms width σ_i we wish to measure, the number of events required to achieve a specified accuracy, $\delta_{\sigma_i}/\sigma_i$ varies as the fourth power of the ratio σ_D/σ_i.

Thus there is a severe statistical penalty unless

$$\sigma_D < \sigma_i. \tag{10}$$

However, once this relation (10) is satisfied,

$$\frac{\delta_{\sigma_i}}{\sigma_i} \approx \sqrt{\frac{1}{2N}}. \tag{11}$$

In this case, about 10,000 measurements would be required to reach a 1% relative uncertainty in σ_i

If, say, only 1% of the beam muons occupy the relevant part of phase space, a typical run would require measurement of 10^6 muons.

[1] See sec. 2.2 of the chapter on *Probability, Statistics and Monte Carlo* of the *Review of Particle Properties*.

2.2 Large-N Limit

In the other limit that counting statistics, but not detector resolution, can be neglected, we have

$$\frac{\delta_{\sigma_i}}{\sigma_i} = \left(\frac{\sigma_D}{\sigma_i}\right)^2 \frac{\delta_{\sigma_D}}{\sigma_D} = \frac{\sigma_D}{\sigma_i}\frac{\delta_{\sigma_D}}{\sigma_i}. \tag{12}$$

The second form of eq. (12) tells us that the uncertainty in the rms width σ_i can not be less than the ratio σ_D/σ_i times the uncertainty in the detector resolution. Good results can only be obtained if σ_D/σ_i is less than one, and if this ratio is much less than one very good results are possible.

2.3 Maximum Acceptable Detector Resolution

To achieve the goal of measurement accuracy $\delta_{\sigma_i}/\sigma_i = 0.01$, we require that the effect of detector resolution be no more than half in quadrature, *i.e.*, $\delta_{\sigma_i}/\sigma_i < 0.007$ as the number of measurements grows large.

We also suppose that the uncertainty in the detector resolution function will be no more than 20%:

$$\frac{\delta_{\sigma_D}}{\sigma_D} < 0.2. \tag{13}$$

Then eq. (12) tells us that the detector resolution must obey

$$\sigma_D < \sqrt{\frac{\delta_{\sigma_i}/\sigma_i}{\delta_{\sigma_D}/\sigma_D}}\sigma_i = 0.19\sigma_i. \tag{14}$$

The first part of expression (14) indicates that if we know the detector resolution function to the same accuracy as we desire for $\delta_{\sigma_i}/\sigma_i$ then the detector resolution can be the same as σ_i. In particular,

$$\text{If} \quad \frac{\delta_{\sigma_D}}{\sigma_D} < 0.01, \qquad \text{then we can have} \qquad \sigma_D \approx \sigma_i \qquad \text{and} \qquad \frac{\delta_{\sigma_i}}{\sigma_i} = 0.01. \tag{15}$$

3 Implications

We now consider the specific implications of the previous analysis to measurements of the various phase-space projections. Table 1 lists various parameters of the phase space to be explored in the cooling experiment.

The criterion (14) then requires the detector resolution to have values listed in Table 2. The more demanding requirements are on the momentum and time measurements. Supposing the bend angle in the momentum-analysis dipole is $\theta_x \approx 1$ radian, then we can use the relation $\sigma_P/P = \sigma_\theta/\theta$ to convert the requirement on $\sigma_{P,D}$ to one on the resolution in angle, namely $\sigma_{x',D} = 6$ mrad. This is only slightly stronger than the direct requirement on $\sigma_{x',D}$.

Table 1: Phase-space parameters of the FOFO-channel cooling experiment.

Parameter	Input Value	Output Value
P (MeV/c)	165	165
E (MeV)	198	198
γ	1.85	1.85
β	0.84	0.84
$\gamma\beta$	1.56	1.56
$\epsilon_{x,N} = \epsilon_{y,N}$ (π mm-mrad)	1200	600
$\epsilon_x = \epsilon_y$ (π mm-mrad)	769	385
$\sigma_x = \sigma_y$ (mm)	10	10
$\sigma_{x'} = \sigma_{y'}$ (mrad)	77	39
σ_P/P	0.03	0.04
$\sigma_E/E = \beta^2\sigma_P/P$	0.021	0.028
σ_t (cm)	1	1.2
$\sigma_t = \sigma_t/\beta c$ (ps)	40	48

Table 2: Required detector resolution to achieve measurement accuracy of 1% on the rms widths σ_i, assuming the detector resolution function is known to 20%, *i.e.*, $\delta_{\sigma_D}/\sigma_D = 0.2$. According to eq. (14), the required detector resolution σ_D varies as the reciprocal of the square root of the uncertainty δ_{σ_D} in the resolution. The requirement on the momentum resolution $\sigma_{P,D}/\sigma_P$ leads to a second requirement on the angular resolution $\sigma_{x',D}$.

Parameter	Value
$\sigma_{x,D} = \sigma_{y,D}$	2 mm
$\sigma_{x',D} = \sigma_{y',D}$	8 mrad
$\sigma_{P,D}/P$	0.006
[$\Rightarrow \sigma_{x',D}$	6 mrad]
$\sigma_{z,D}$	2 mm
$\sigma_{t,D}$	8 ps