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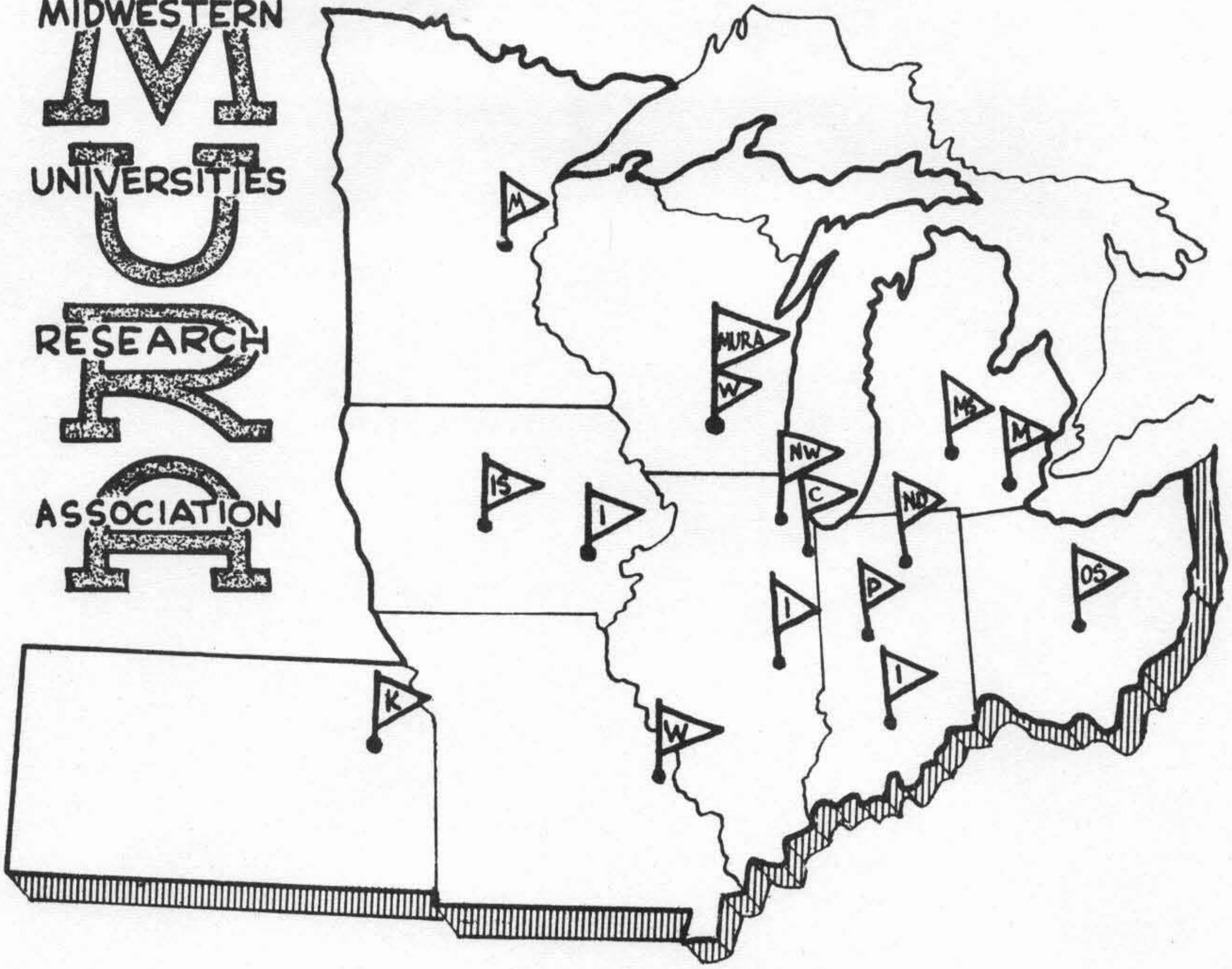

MODIFICATION OF LIOUVILLE'S THEOREM REQUIRED BY THE PRESENCE OF DISSIPATIVE FORCES

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MODIFICATION OF LIOUVILLE'S THEOREM REQUIRED BY THE PRESENCE OF DISSIPATIVE FORCES*

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It has recently been suggested by O'Neill[1] that high current densities might be achieved in accelerators by the use of foils to reduce the volume in phase space occupied by a beam of particles. It is the purpose of this note to examine under what conditions such a compression of phase space can occur and whether the effect is large enough to be of any practical value in accelerators.

The equations of motion satisfied by a particle can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

where the Lagrangian L includes all forces derivable from a potential, the Q_i are the forces due to the foil and the q_i are the generalized coordinates of the particle.

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1. G. K. O'Neill, Phys Rev 102 1418 (1956).

If we define

$$P_i = \frac{\partial L}{\partial \dot{q}_i},$$

$$H = \sum_i p_i \dot{q}_i - L$$

we get

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + Q_i \end{aligned}\right\} \tag{1}$$

We now consider a closed region V in phase space. The rate of change of this volume will be equal to the volume integral of its divergence:

$$\frac{dV}{dt} = \int \pi_i \, dq_i \, dp_i \sum_i \left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} \right). \tag{2}$$

Using Eq. (1), Eq. (2) becomes

$$\frac{dV}{dt} = \int \pi_i \, dq_i \, dp_i \sum_i \frac{\partial Q_i}{\partial p_i}. \tag{3}$$

Therefore to see what happens to a volume in phase space due to a foil, we need merely consider the form of the functions Q_i. Of special interest is the case of an ideal foil, defined as one which produces an energy loss but no scattering. If, furthermore, the energy loss depends on the path length through the foil, but not on the particle velocity, we may write

$$\underset{\sim}{Q} = Q(\underset{\sim}{r}, t) \frac{\dot{\underset{\sim}{r}}}{\dot{r}} = Q(\underset{\sim}{r}, t) \frac{(\underset{\sim}{p} - e\underset{\sim}{A}/c)}{|\underset{\sim}{p} - e\underset{\sim}{A}/c|}$$

where $\underset{\sim}{r}$ is the position of the particle and $\underset{\sim}{A}$ is the vector potential. By writing the time t explicitly, we take into account that the foil need not remain in one position. Note that $Q(\underset{\sim}{r}, t)$, is negative for a foil. With the above choice of Q we obtain

$$\sum_i \frac{\partial Q_i}{\partial p_i} = 2\,\frac{Q(\underset{\sim}{r},t)}{P} \tag{4}$$

where $P = |\underset{\sim}{p} - e\underset{\sim}{A}/c|$ is the kinetic momentum of the particle. The factor 2 comes from the fact that we are considering the problem in three dimensions. If the effect of the foil on vertical oscillations is neglected, the factor is unity. Using Eq. (4), Eq. (3) becomes

$$dV/V = 2\,dt\,\overline{Q(\underset{\sim}{r},t)/P} \tag{5}$$

where the bar indicates the space average of Q/P.

In most accelerators, the momentum spread of the beam is much smaller than the average momentum P of the particles. Therefore instead of Eq. (5) we may write

$$dV/V = (2/P)\,dP \tag{6}$$

where $dP = \overline{Q}\,dt$ is the average momentum increment of a particle due to the foils in time dt. Integrating, we obtain, if the foils are the only source of momentum increment,

$$\frac{V_f}{V_i} = \left(\frac{P_f}{P_i}\right)^2 \tag{7}$$

where the subscript i indicates the initial value and f the final value. It is apparent from Eq. (7) that the volume in phase space may be reduced by the use of an ideal foil, but that the average momentum of the particles must be reduced by an amount comparable to the reduction in phase space. To avoid this, an oscillator can be used to supply the energy lost in the foil. Then the average momentum P is kept constant and, on integrating Eq. (6), we get

$$\frac{V_f}{V_i} = e^{-2\Delta P/P_i} \tag{8}$$

where Δ P is the average total momentum loss in the foil.

An actual foil differs from an ideal foil in that the energy loss of a particle depends on the magnitude of the particle momentum. However, in the relativistic region this dependence is small and can be neglected so that Eqs. (7) and (8) still approximately hold.

In the non-relativistic region the energy loss goes approximately as the inverse square of the velocity so that the force becomes

$$\underset{\sim}{Q} = Q(x,t)\,\underset{\sim}{P}/P^3.$$

Putting this expression for Q in Eq. (3) it turns out that $dV/dt = 0$. From Eqs. (7) and (8) it is apparent that the reduction of the volume in phase space depends only on the energy loss in the foil. Therefore, although foils of odd shapes and those which change with time may twist a volume in phase space, they are no more effective in reducing the volume than are uniform foils which produce the same average energy loss. In order to increase the density of particles in the beam by a factor n. a reduction in phase volume by a factor n is required, which from Eq. (8) implies a loss of momentum to the foil of

$$\Delta P = \frac{P}{2} \operatorname{Log} n$$

which is comparable to P itself. An actual foil thickness sufficient to do this would produce more than enough scattering to cancel the compression in phase space obtained above.