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Physics analysis performed by ECALC9

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Gregg Penn's program ECALC9 has become the de facto emittance calculation standard for the Muon Collaboration. We give here a logical presentation of the physics calculations that are made in the program. This is done in order to (1) document how the cooling performance in ICOOL simulations was computed, (2) to help non-ICOOL users adopt the code to other programs, and (3) make all users more aware of what the program actually does.

1. Introduction

The physics analysis program ECALC9 has become a de facto standard in the Muon Collaboration for computing the emittance and related quantities in solenoidal focusing channels. The current version 2 of the physics analysis in the program was written by Gregg Penn in June 2001. The program was originally written to use with ICOOL. A modified version ECALC9F that takes its control parameter input from a file instead of the keyboard has been distributed as part of the standard ICOOL distribution for several years.

Recently users of other programs have expressed interest in using the same emittance analysis as ICOOL. This report contains a logical presentation of the calculations performed in ECALC9. Hopefully this document will make the conversion task to other programs somewhat easier. A second and more important reason is that people should not accept the results of this (or hopefully any other) program without some understanding of what it actually does. It is important to spell out exactly what is being calculated and where the various cuts in the program are actually applied.

2. Program structure

Calculations in the program are controlled with data from the file ECALC9F.INP with input variables given in Table 1.

Table 1: Variables defined in ECALC9F.INP

variable	meaning	units
USEPTYPE	particle type	2 = muon
PZMIN	minimum p_Z in analysis	GeV/c
PZMAX	maximum p_Z in analysis	GeV/c
TRANSCUTA	transverse acceptance cut	m rad
TRANSCUTB	2^{nd} transverse acceptance cut	m rad
LONGCUT	longitudinal acceptance cut	m rad
RFFREQ	RF frequency	MHz
SIGMA_CUT	standard deviations for acceptance cut	#
PZCORR	correct for transverse amplitude-momentum correlation?	(logical)

The main program has an outer loop over geometry regions, which have constant values of the longitudinal coordinate z. There is an inner loop over particles, which is used to exclude some particles from further analysis and to redefine the time variable. The variables read in from the output of a tracking program (e.g. FOR009.DAT for ICOOL) are

1. region number
2. B_Z on-axis
3. x, y, z
4. p_X, p_Y, p_Z
5. time

The maximum number of input particles is 100K. Particles are immediately rejected if

1. error flag is not zero
2. particle type is not USEPTYPE
3. this is the reference particle
4. p_Z < PZMIN
5. p_Z > PZMAX

The time variable for each particle is changed from the absolute time used by ICOOL to time *t* relative to the reference particle

$$t = t_{abs} - t_{ref}$$

If RFFREQ (*f*) is given, the time is further adjusted modulo the period of the RF wave.

This is done using the pseudocode

```
count = 0.5 + t  f
if( count < 0 ) then    count = count – 1
t = t  -  INT(count)  /  f
```

Once the particle loop is completed, the physics quantities for this region are computed by calling the subroutine EMIT.

3. Subroutine EMIT

The numbered equations in this section are the computed variables that are written out to ECALC9F.DAT at the completion of each call to this subroutine.

3.1 First loop over particles

The first loop in EMIT is mainly used to compute transverse quantities. The expectation value of a quantity *q* is defined as

$$\langle q \rangle = \frac{\sum_{i=1}^{N} q_i w_i}{\sum_{i=1}^{N} w_i}$$

where w_i is the weight. ECALC9 first uses the tracking variables $\{x, y, t, p_X, p_Y, E\}$. The quantities

$$\langle x \rangle \qquad \langle y \rangle \qquad \langle p_Z \rangle \tag{1-3}$$

are computed for output. The 6-dimensional covariance matrix *M* is computed from the six tracking variables

$$M_{i,j} = \langle q_i q_j \rangle - \langle q_i \rangle \langle q_j \rangle$$

The standard deviations of the time and energy follow directly

$$\sigma_t = \sqrt{M_{3,3}} \qquad \sigma_E = \sqrt{M_{6,6}} \tag{4-5}$$

The dispersions are calculated from

$$
\begin{aligned}
D_X &= M_{1,6} \frac{\langle E \rangle}{M_{6,6}} \\
D_Y &= M_{2,6} \frac{\langle E \rangle}{M_{6,6}} \\
D_R &= \left(\langle ER \rangle - \langle E \rangle \langle R \rangle \right) \frac{\langle E \rangle}{M_{6,6}} \\
D_{R^2} &= \left(\langle ER^2 \rangle - \left(M_{1,1} + M_{2,2} \right) \langle E \rangle \right) \frac{\langle E \rangle}{M_{6,6}}
\end{aligned}
\qquad (6\text{-}9)
$$

where

$$R = \sqrt{x^2 + y^2}$$

The 6-dimensional normalized emittance is

$$\varepsilon_{6N} = \frac{c}{m^3} \sqrt{\det(M)} \qquad (10)$$

The transverse normalized emittance is found by computing a second covariance matrix M^T made up from the four tracking variables $\{x, y, p_X, p_Y\}$.

$$\varepsilon_{TN} = \frac{1}{m} \sqrt{\sqrt{\ } \det\left(M^T\right)} \qquad (11)$$

Define the solenoidal strength parameter

$$\kappa = \frac{eB_Z}{2\langle p_Z \rangle}$$

and the mechanical angular momentum

$$L_m = \langle xp_Y \rangle - \langle yp_X \rangle$$

Then the canonical angular momentum is

$$L_c = L_m + \frac{eB_Z}{2} \left(\langle x^2 \rangle + \langle y^2 \rangle \right) \qquad (12)$$

The dimensionless canonical angular momentum is

$$L_d = \frac{\left[M_{1,5} - M_{2,4} + \frac{eB_Z}{2} \left(M_{1,1} + M_{2,2} \right) \right]}{2m\varepsilon_{TN}} \tag{13}$$

The Courant-Snyder transverse beta function is

$$\beta_T = \frac{\left(M_{1,1} + M_{2,2} \right) \langle p_Z \rangle}{2m\varepsilon_{TN}} \tag{14}$$

while the transverse alpha function is

$$\alpha_T = -\frac{\left(M_{1,4} + M_{2,5} \right)}{2m\varepsilon_{TN}} \tag{15}$$

and the transverse gamma function is

$$\gamma_T = \frac{M_{4,4} + M_{5,5}}{2m\varepsilon_{TN} \langle p_Z \rangle}$$

3.2 Second loop over particles

The second loop in EMIT is used to compute longitudinal quantities. First transverse particle variables are corrected for any mean values

$$x_c = x - \langle x \rangle$$

and so forth. The transverse amplitude (squared) for each particle is computed from

$$A_T^2 = \frac{1}{m} \left[\frac{\beta_T}{\langle p_Z \rangle} \left(p_{Xc}^2 + p_{Yc}^2 \right) + \gamma_T \langle p_Z \rangle \left(x_c^2 + y_c^2 \right) + 2\alpha_T \left(x_c p_{xc} + y_c p_{yc} \right) + 2\left(\beta_T \kappa - L_d \right) \left(x_c p_{yc} - y_c p_{xc} \right) \right]$$

A third covariance matrix M^L is computed using the three variables $\{t, E, A_T^2\}$. There are then two branches, depending on how the user has set the parameter PZCORR.

3.2.1 PZCORR = false

This branch is used when the correlation between momentum and transverse amplitude is not taken into account. The longitudinal normalized emittance is

$$\varepsilon_{LN} = \frac{c}{m}\sqrt{M^L_{1,1}M^L_{2,2} - \left(M^L_{1,2}\right)^2} \qquad (16)$$

The longitudinal alpha function is

$$\alpha_L = \frac{cM^L_{1,2}}{m\varepsilon_{LN}} \qquad (17)$$

We also define

$$\delta = \frac{cM^L_{1,1}}{m\varepsilon_{LN}}$$

The correlation parameters C_t and C_E, defined below, are 0 in this case.

3.2.2 PZCORR = true

This branch is used when the correlation between momentum and transverse amplitude is taken into account. The longitudinal normalized emittance is

$$\varepsilon_{LN} = \frac{c}{m}\frac{\sqrt{\det\left(M^L\right)}}{M^L_{3,3}} \qquad (16')$$

The longitudinal alpha function is

$$\alpha_L = \frac{c}{m\varepsilon_{LN}}\left(M^L_{1,2} - \frac{M^L_{1,3}M^L_{2,3}}{M^L_{3,3}}\right) \qquad (17')$$

The delta function is

$$\delta = \frac{c}{m\varepsilon_{LN}}\left(M^L_{1,1} - \frac{\left(M^L_{1,3}\right)^2}{M^L_{3,3}}\right)$$

The correlation parameter for the time is

$$C_t = \frac{M^L_{1,3}}{M^L_{3,3}} \tag{18}$$

while the correlation parameter for the energy is

$$C_E = \frac{M^L_{2,3}}{M^L_{3,3}} \tag{19}$$

In both cases the longitudinal beta function is defined as

$$\beta_L = \frac{c\delta\langle p_Z\rangle^3}{m^2 + \langle p_Z\rangle^2} \tag{20}$$

The standard deviation of the energy distribution is also computed a second way, taking the momentum-transverse amplitude correlation into account

$$\sigma_{Ec} = \sqrt{M^L_{2,2} - C_E{}^2 M^L_{3,3}} \tag{21}$$

3.3 Third loop over particles

The third loop in EMIT is used to compute particle densities in fixed phase space volumes. The total number of accepted particles is

$$n_0 = \sum_{i=1}^{N} w_i \tag{22}$$

The position and momentum vectors are corrected for their mean values, as above in section 3.2. A corrected time variable is computed for each particle from

$$t_c = t - \langle t\rangle - C_t\left(A_T^2 - \langle A_T^2\rangle\right)$$

Likewise, a corrected energy variable is computed for each particle from

$$E_c = E - \langle E\rangle - C_E\left(A_T^2 - \langle A_T^2\rangle\right)$$

The longitudinal amplitude is

$$A_L = \frac{c}{m}\left[\frac{t_c^2}{\delta} + \delta\left(E_c - \frac{\alpha_L t_c}{\delta}\right)^2\right]$$

The code now makes the test: if A_T^2 < TRANSCUTA and A_L < LONGCUT, include this event weight in the sum

$$n_1 = \sum_{i=1}^{K_1} w_i \qquad (23)$$

where K_1 is the number of events that satisfies the cuts. Similarly, a second transverse cut is used for the test: if A_T^2 < TRANSCUTB and A_L < LONGCUT, include this event weight in the sum

$$n_2 = \sum_{i=1}^{K_2} w_i \qquad (24)$$

3.4 Tail-cutting logic

If the user has set the SIGMA_CUT parameter (n_σ), the subroutine can iterate up to 20 times, rejecting particles in the tails of the transverse and longitudinal amplitude distributions. There is a fourth loop over particles. The particle is rejected if

$$A_T^2 > 2n_\sigma^2 \varepsilon_{TN}$$

or

$$A_L > n_\sigma^2 \varepsilon_{LN}$$

and the number N of currently active particles is adjusted accordingly. All the calculations in EMIT are then repeated until no additional tail particles are rejected.