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Behavior of Kinematic Invariants of Beams with Large Energy Spread

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Abstract

The well known kinematic moment invariants, emittance $\epsilon_N = \mathbf{det}(\Sigma)$ and $I_N = -0.5\mathbf{Tr}(J\Sigma J\Sigma)$ [1] in linear beam dynamics are computed using ICOOL [2] for a simple drift problem. We show that they are not constant for a non-paraxial beam with significant energy spread (> 5%). This fact may have implications for the MICE experiment.

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I. INTRODUCTION: THE PROBLEM

All those running the code ICOOL have noticed *peculiar* behavior of the transverse emittance (ϵ_T) in several problems. Most recently in a study of the MICE cooling channel, Bob Palmer pointed out that the emittance, in the region of no-absorbers, should have been constant; however, the ICOOL runs show a decrease between the first and second absorber and that the same behavior is shown in an identical channel without absorber and rf.

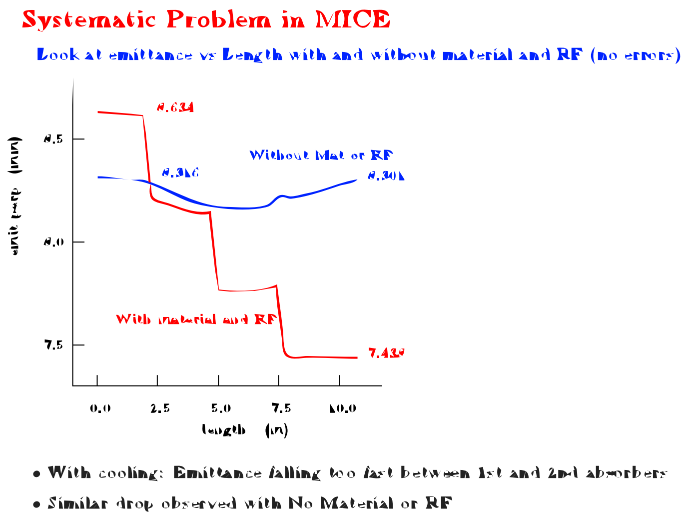

FIG. 1: MICE cooling channel. The transverse emittance does not remains constant in the region between the absorbers, as naively was expected. The superimposed curve depicts the emittance of the channel without material; both curves shows similar behavior. (color)

Afterward R. Fernow look at an even simpler problem: a plain **DRIFT** and to our surprise $\epsilon_\mathbf{T}, \epsilon_\mathbf{L}, \epsilon_\mathbf{6}$ changed although one expected that at least ϵ_6 should remain constant. To be precise, *the emittance as calculated by* **ECALC9** *is not constant.*

II. THEORY

It is well know that under linear symplectic (canonical, hamiltonian) transformations there are a number of **kinematic moment invariants**. In particular, of interest and

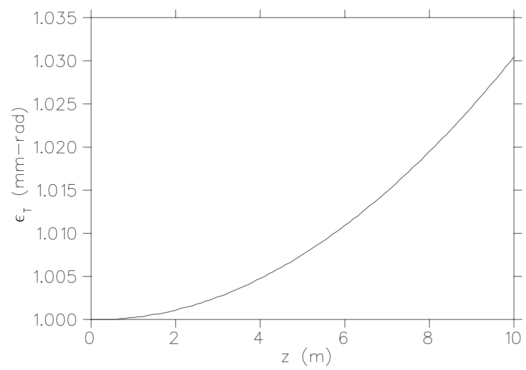

FIG. 2: ϵ_T *vs.* z as calculated by ICOOL and reported by ECALC9 for a 10 m drift for a Gaussian beam. (color)

widely used, the quadratic invariants are:

$$\epsilon_N = \mathbf{det}(\Sigma) \quad \text{the invariant 6D emittance and} \tag{1}$$

$$I_N = -0.5\mathbf{Tr}(J\Sigma J\Sigma) \tag{2}$$

This second invariant is not used much in practice, although has been in the literature for many years [1], because it has not been possible to assign a geometric interpretation to it. We only notice that given a 6×6 matrix Σ constructed with the second order moments of the beam distribution, we can obtain a number (a scalar) by taking the determinant or the trace of it.

In the equation above J is the unit symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{3}$$

where I is the unit matrix in 3D space. The calculation of I_N has been implemented in *ecalc9f.for*

III. POSSIBLE SOLUTION

Subsequently, we learned of the paper: K. Floettmann [3], where in the abstract, the blasphemous statement is made: *the [transverse] emittance of a beam is not necessarily constant in a drift space.* The author argues that energy spread is responsible for this behavior. The proof given is not complicated but laborious.

The normalized emittance is: $\epsilon_x^{norm} = \frac{1}{m_o c}\sqrt{< x^2 >< p_x >^2 - < x p_x >^2}$. In a drift $x(s) = x(0) + \frac{p_x}{p_s} s$ and $p_x(s) = p_x(0)$; hence

$$\frac{1}{N^2}\sum x(0)\frac{p_x}{p_s}\sum p_x^2 \text{ must be } = \frac{1}{N^2}\sum x(0)p_x \sum \frac{p_x}{p_s}p_x \tag{4}$$

and

$$\frac{1}{N^2}\sum \left(\frac{p_x}{p_s}\right)^2 \sum p_x^2 \text{ must be } = \frac{1}{N^2}\left(\sum \frac{p_x}{p_s}p_x\right)^2. \tag{5}$$

which is true only if the energy spread is very small.

This we may accept because one argues that ϵ_6 is the real constant of motion and ϵ_T may change as long as ϵ_L changes too to keep ϵ_6 constant.

So let us see what ECALC9 is saying about ϵ_6 in the drift example:

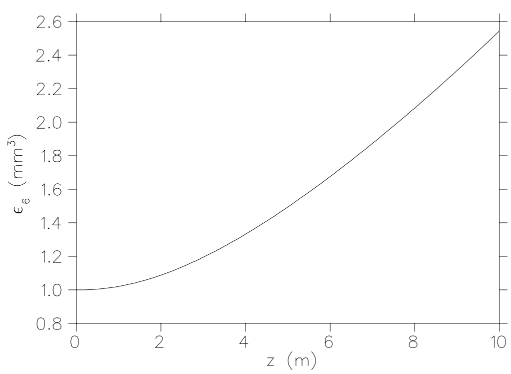

FIG. 3: ϵ_6 *vs.* z as calculated by ICOOL and reported by ECALC9 for a 10 m drift for a Gaussian beam.(color)

As a function of the energy spread

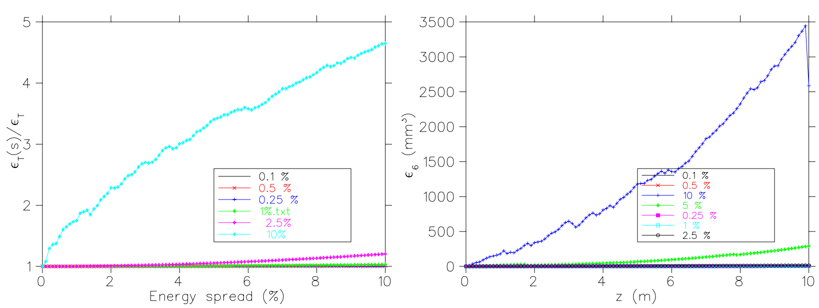

FIG. 4: ϵ_T and ϵ_6 *vs.* z as calculated by ICOOL and reported by ECALC9 for a 10 m drift for a Gaussian beam; the different lines refers to different energy spreads. (color)

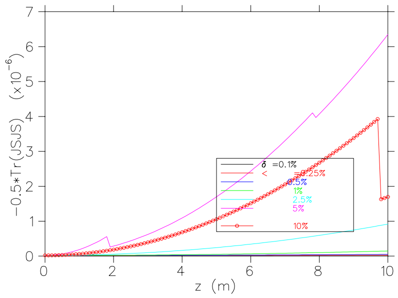

FIG. 5: I invariant *vs.* z as calculated by ICOOL and reported by ECALC9 for a 10 m drift for a Gaussian beam. (color)

IV. CONCLUSION

As was demonstrated by J.S. Berg [4] the seemingly simple **DRIFT** problem reveals its true non-linear features when we have a non-paraxial beam with substantial energy spread.

[1] G. Rangarajan, F. Neri and A. Dragt, *Generalized emittance invariants* Proc. of the 1989 PAC, Chicago, IL pp. 1280 and references there.

[2] R. Fernow, *ICOOL: A simulation code for Ionization Cooling of Muon Beam*, Proc. Particle Accelerator Conference, 1999.

[3] K. Floettmann, *Some basic Features of the Beam Emittance*, Phys. Rev. ST Accel. Beams **6**, 034202 (2003)

[4] J.S. Berg, *Understanding emittance growth in drifts*, http://www.cap.bnl.gov/mumu/conf/E-EX-040121/berg-E-EX-1.pdf