Mathematical Model Analysis for Hg Flow

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April 20th, 2010

Outline

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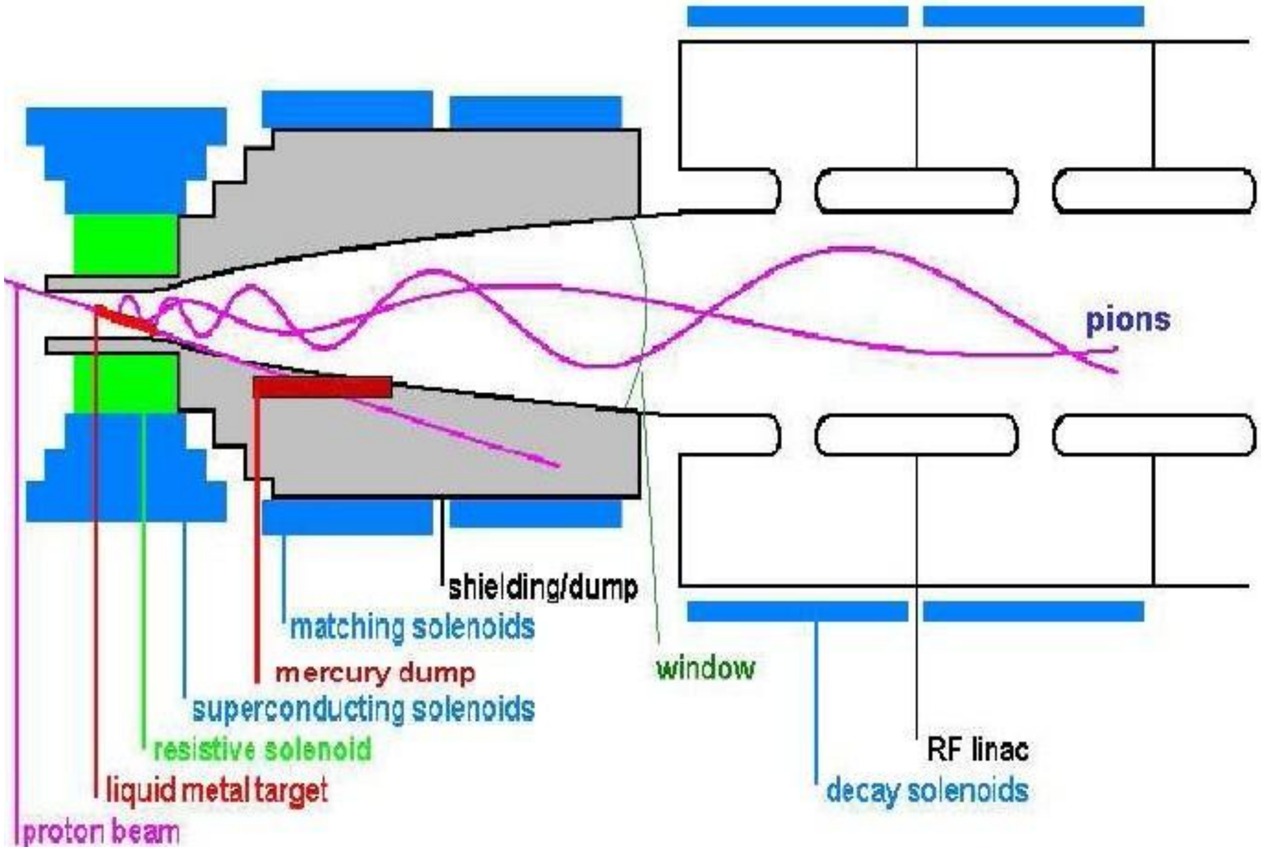
  - Incompressible Flow (MHD)
  - Proton Beam Energy Conversion
  - Boundary Condition Discussion
- Equation of State

Introduction

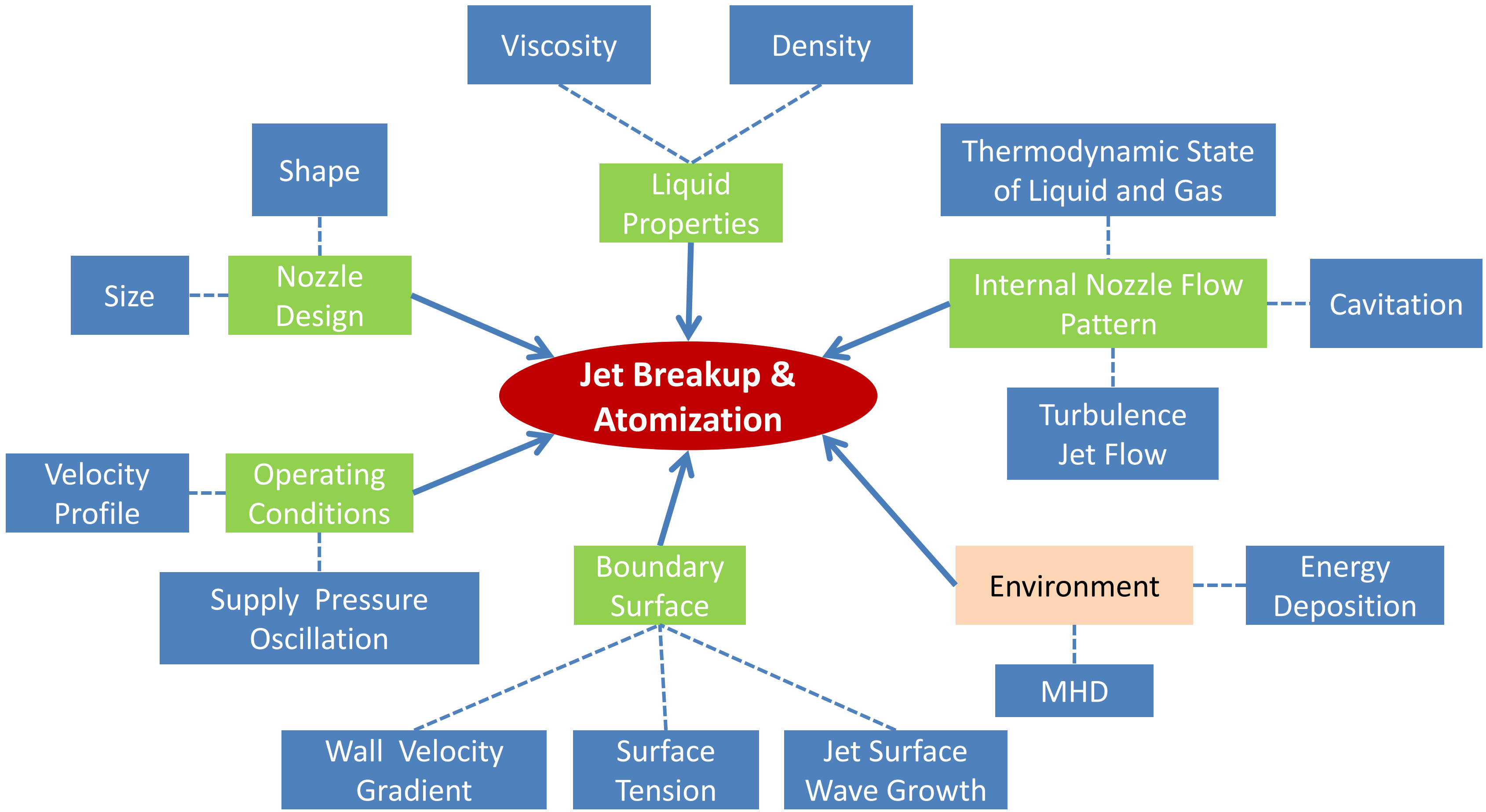


Neutrino Factory/Muon Collider

The shape of the jet is important!

K.T. McDonald, “The R&D Program for target and capture at a Neutrino Factory and Muon Collider Source”, NF&MCC Technical Board Meeting, LBL, Oct.3, 2000

Introduction—Mechanism of Jet Breakup & Atomization



R.D. Reitz and F.V. Bracco, Mechanism of atomization of a liquid jet,Phys. Fluids, Vol.25, No.10,Oct.1982

Introduction—Dynamic Problems in Hg Target Flow

- Internal Flow
  - Dynamics of MHD pipe flow
  - The influence of nozzle design on jet exit conditions
- Jet
  - Dynamics of free jet under MHD & energy deposition
  - Jet breakup & instability mechanism and the effects of MHD & energy deposition
  - Comparison with known classical free jet dynamics
  - Inlet conditions to the Dump

Mathematical Model for Hg Flow —Parameters (1)

Mercury Properties (25 ^{0}C)	
Density	13.546 kg/L
Sound Speed	1451 m/s
Viscosity	1.526E-3 kg/s·m
Dynamic Viscosity	1.127 m^2/s
Thermal Conductivity	8.69 W/m·K
Electrical Conductivity	1E6 Siemens/m
Specific Heat	0.139 J/kg·K
Prandtl Number	0.025
Surface Tension	465 dyne/cm
Permeability	4*PI*1E-7

Variable Values	
Velocity in Pipe	3.4 m/s
Beam Energy	24 GeV
Beam Intensity	10 TP
Beam Pulse Length	2 ns
Magnetic Strength	20 Tesla
Pipe Diameter	2.54 cm (1″)
Jet Diameter	1 cm
Beam Diameter	0.15 cm RMS
Static Pressure inside Pipe	18.5 Bar (kPa)
Dynamic Pressure inside Pipe	0.7 Bar (kPa)
Air Density Outside Jet	0.0013 kg/L
Jet Tilts w.r.t. Magnetic Axis	33 mrad
Beam Tilts w.r.t. Magnetic Axis	67 mrad

Mathematical Model for Hg Flow —Parameters (2)

$$R_{e_{pipe}} = \frac{\rho v_{pipe} d_{pipe}}{\mu} = \frac{13.6\times10^3\times20\times0.0254}{1.5\times10^{-3}} = 7.666\times10^5$$

$$R_{e_{mag,pipe}} = \beta v_{jet} d_{jet} \sigma_{Hg} = 4\times\pi\times10^{-7}\times3.4\times0.0254\times1\times10^6 = 0.108$$

$$M_{a_{pipe}} = \frac{v_{pipe}}{v_{sound}} = \frac{3.4}{1451} = 2.343\times10^{-3}$$

$$R_{e_{jet}} = \frac{\rho v_{jet} d_{jet}}{\mu} = \frac{13.6\times10^3\times20\times0.01}{1.5\times10^{-3}} = 1.8\times10^6$$

$$R_{e_{mag,jet}} = \beta v_{jet} d_{jet} \sigma_{Hg} = 4\times\pi\times10^{-7}\times20\times0.01\times1\times10^6 = 0.2512$$

$$M_{a_{jet}} = \frac{v_{jet}}{v_{sound}} = \frac{20}{1451} = 0.014$$

$$W_{e_{jet}} = \frac{\rho d_{jet} v_{jet}^2}{\tau} = \frac{13.6\times10^3\times0.01\times20^2}{465\times10^{-3}} = 115000$$

Small Ma → Incompressible

Re* for Curved Pipe?
Re* effected by B Field?
} → Turbulence ?

(Re* for Curved Tube is Larger than Straight Pipe*)

Mathematical Model for Hg Flow —Incompressible Flow (No MHD)

Governing Equations

$$\nabla \cdot \vec{V} = 0$$

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}\right) = -\nabla P + \mu\nabla^2\vec{V} + \rho\vec{g}$$

$$\rho C_p\frac{dT}{dt} = \frac{DP}{Dt} + \nabla\cdot(k\nabla T) + \Phi$$

Where $\Phi = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Nondimensional Governing Equations

$$\nabla^* \cdot \vec{V}^* = 0$$

$$\rho^*\frac{D\vec{V}^*}{Dt^*} = -\nabla^* P^* + \frac{1}{Re}\nabla^*\cdot\left(\frac{\partial u_i{}^*}{\partial x_j{}^*} + \frac{\partial u_j{}^*}{\partial x_i{}^*}\right) + \frac{1}{Fr}\rho^*$$

$$\rho^*\frac{DT^*}{Dt^*} = Ec\frac{DP^*}{Dt^*} + \frac{1}{PrRe}\nabla^*\cdot(k^*\nabla^* T^*) + \frac{E_c}{R_e}\Phi^*$$

$V, r, t, \rho, P, \mu, g, C_p, \Phi, T, k$ (11)

$d_0, V_0, \rho_0, \mu_0, g, C_p, \Delta T$ (7)

R_e, Fr, Ec, Pr (4)

Reynolds Number $R_e = \frac{\rho_0 V_0 d_0}{\mu_0}$

Froude Number $Fr = \frac{V_0}{\sqrt{d_0 g}}$

Mach Number $M_a = \frac{V_0}{c}$

Magnetic Reynolds Number $Re_{mag} = \beta_0 V_0 d_0 \sigma_0$

Eckert Number $Ec = \frac{V_0{}^2}{C_p \Delta T} = (\gamma - 1)Ma^2$

Prandtl Number $Pr = \frac{\mu_0}{\rho_0 k_0}$

Mathematical Model for Hg Flow — Incompressible Flow (MHD)

Governing Equations

$$\nabla \cdot \vec{V} = 0$$

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}\right) = -\nabla P + \mu \nabla^2 \vec{V} + \rho \vec{g} + \boxed{\vec{J} \times \vec{B}}$$

Lorentz Force

$$\rho C_p \frac{DT}{Dt} = \frac{DP}{Dt} + \nabla \cdot (k \nabla T) + \Phi + \boxed{\frac{1}{\sigma}\vec{J}^2}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) - \nabla \times (\frac{c^2}{4\pi\sigma} \nabla \times \vec{B})$$

$$\vec{J} = \frac{c}{4\pi\beta} \nabla \times \vec{B}$$

$$\nabla \cdot \vec{B} = 0$$

Nondimensional Governing Equations

$$\nabla^* \cdot \vec{V}^* = 0$$

$$\rho^* \frac{D\vec{V}^*}{Dt^*} = -\nabla^* P^* + \frac{1}{Re} \nabla^* \cdot \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*}\right) + \vec{J}^* \times \vec{B}^* + \frac{1}{Fr} \rho^*$$

$$\rho^* \frac{DT^*}{Dt^*} = Ec \frac{DP^*}{Dt^*} + \frac{1}{PrRe} \nabla^* \cdot (k^* \nabla^* T^*) + \frac{E_c}{R_e} \Phi^* + \frac{R_e^2 E_c}{R_{e_{mag}}} \vec{J}^{*2}$$

$$\frac{\partial \vec{B}^*}{\partial t} = \nabla^* \times (\vec{V}^* \times \vec{B}^*) - \frac{1}{Re_{mag}} \nabla^{*2} \vec{B}^*)$$

$$\vec{J}^* = \frac{c}{4\pi} \frac{1}{Re} \nabla^* \times \vec{B}^*$$

$$\nabla^* \cdot \vec{B}^* = 0$$

$V, r, t, \rho, P, \mu, g, J, B, C_p, \Phi, T, k, \sigma, c, \beta$ **(16)**

↓ $d_0, V_0, \rho_0, \mu_0, g,$ $C_p, \Delta T, \sigma_0, c, \beta_0, k_0$ **(11)**

$R_e, R_{e_{mag}}, Fr, Ec, Pr$ **(5)**

Mathematical Model for Hg Flow — Proton Beam Energy Conversion

- Assume as a δ function (A. Hassanein)

$$E_{pulse}(r,t) = q_b\delta(t-t_b) = q_0 exp(-(r/r_b)^2)\delta(t-t_b)$$ where $q_0 = \frac{Q_b}{\pi r_b^2}$

- Instantaneous energy deposition (P. Sievers and P. Pugnat)

$$E \to \Delta T_0 \xrightarrow{P=\kappa\alpha_V\Delta T} P \xrightarrow{\frac{dE_c}{dV}=\frac{(\alpha_V\Delta T(r))^2}{2\kappa}} E_c \xrightarrow{\frac{\kappa(\alpha_V\Delta T)^2}{2}=\frac{\rho v^2}{2}} v$$

- 1D parabolic energy distribution of the beam long the propagation direction of the Hg flow. Sin^2 Envelop with τ=2ns pulse length (I.F. Barna et al.)

$$E_{pulse}(x,t) = E_0 sin^2\left(\frac{\pi t}{\tau}\right)(1-(x/x_s)^2)$$

- Monte Carlo Code (MARS, GRAN, FLUKA)

- Using Sergei's calculation for a 24-GeV, 10-TP proton beam (W.Bo, R. Samulyak)

Mathematical Model for Hg Flow — Boundary Conditions

Pipe Flow

- No slip at the pipe wall;
- Non-conducting wall;
- Velocity profile and initial values;
- Inlet and outlet pressure;
- Magnetic field profile;

Jet Flow

- The normal component of the velocity field is continuous across the interface;
- The pressure jump at the interface is defined by the surface tension τ and main radii of curvature:

$$\Delta P_\Gamma = \tau\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \Rightarrow p^* = \frac{1}{\mathrm{W_a}} \frac{s^{*2} + 2s_{\theta^*}^{*2} - s^* s_{\theta^*\theta^*}^*}{(s^{*2} + s_{\theta^*}^{*2})^{3/2}}$$

- The normal component of the current density vanishes at the interface giving rise to the Neumann boundary condition for the electric potential:

$$\frac{\partial \varphi}{\partial \vec{n}_\Gamma} = \frac{1}{c}(\vec{V} + \vec{B}) \cdot \vec{n}$$

- Heat transfer balance at the free surface;

Next Step

- Study the energy deposition conversion for proton beam on jet to set up proper mathematical model for energy equation;
- Jet exit condition:
  - The influence of discontinuity boundary conditions transiting from pipe flow to jet flow;
  - Nozzle design on jet exit condition
- Dynamics of MHD on pipe flow

Equation of State (for compressible flow)

- Continuous Model
- Two Phase EOS
- Homogeneous Model
- Heterogeneous Model
- ISM Model
- SESAME Library
- Summation Method

Equation of State — Continuous Model

- Continuous Model (David P. Schmidt)

$$p = p_l^{sat} + p_{gl} \cdot log\left[\frac{\rho_g \cdot a_g^2 \cdot \left(\rho_l + \alpha \cdot \left(\rho_g - \rho_l\right)\right)}{\rho_l \cdot \left(\rho_g \cdot a_g^2 - \alpha \cdot \left(\rho_g \cdot a_g^2 - \rho_l \cdot a_l^2\right)\right)}\right]$$

$$p_{gl} = \frac{\rho_g \cdot a_g^2 \cdot \rho_l \cdot a_l^2 \cdot \left(\rho_g - \rho_l\right)}{\rho_g^2 \cdot a_g^2 - \rho_l^2 \cdot a_l^2} \quad \alpha = \frac{\rho - \rho_l}{\rho_g - \rho_l}$$

a_g and a_l are the sound speeds of the pure phases. ω is zero in cartesian coordinates and unity in polar coordinates.

Equation of State — Two Phase Model

- Two Phase Model (I.F. Barna et al.)

$$du = c_V\, dT + \left[T\left(\frac{\partial p}{\partial T}\right)_V - p\right] dV.$$

$$c_p = c_V + T\left(\frac{\partial V}{\partial T}\right)_p^2 \left(\frac{\partial p}{\partial V}\right)_T$$

+

Polynomial fits for T Dependence of Cp and αT

→

Internal Energy of Liquid State

+

Vaporization Heat

→

Internal Energy of Liquid State

+

Internal Energy of Gas State

+

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

→

$$\rho_l = \left(\frac{\partial \rho_l}{\partial p}\right)_{u_l} dp + \left(\frac{\partial \rho_l}{\partial u_l}\right)_p du_l$$

$$\rho_g = \left(\frac{\partial \rho_g}{\partial p}\right)_{u_g} dp + \left(\frac{\partial \rho_g}{\partial u_g}\right)_p du_g$$

Equation of State — Homogeneous Model

- Homogeneous Model (Bruggeman)

Liquid/vapor mixture is treated as a pseudo-fluid that obeys an equation of state of a single component flow.

Conductivity of the mixed phase is calculated by

$$\sigma_m = \begin{cases} 0 & \text{if } 1 \geq \beta \geq \frac{2}{3} \\ \frac{1}{2}(2 - 3\beta) & \text{if } \beta \leq \frac{2}{3} \ . \end{cases}$$

Equation of State — Heterogeneous Model

- Heterogeneous Model (Jian Du, Roman Samulyak)

(1) Pure Vapor (Polytropic EOS): $P = (\gamma_v - 1)E\rho$

(2) Liquid (Stiffened Polytropic EOS):

Stiffened EOS Model (incompressible single Liquid Phase)

$$P + \gamma P_\infty = (\gamma - 1)\rho(\epsilon + E_\infty)$$

Adiabatic exponent γ=3.19;
Stiffening constant P_∞=3000bar;
Energy translation E_∞=4.85×10E4 erg/sgK

(3) Liquid-Vapor Mixture:

$$P = P_{sat,l} + P_{vl}\log\left[\frac{\rho_{sat,v}{a_{sat,v}}^2(\rho_{sat,l} + \beta(\rho_{sat,v} - \rho_{sat,l}))}{\rho_{sat,l}(\rho_{sat,v}{a_{sat,v}}^2 - \beta(\rho_{sat,v}{a_{sat,v}}^2 - \rho_{sat,l}{a_{sat,l}}^2))}\right]$$

Where $\beta = \dfrac{\rho - \rho_{sat,l}}{\rho_{sat,v} - \rho_{sat,l}}$

$$P_{vl} = \frac{\rho_{sat,v}{a_{sat,v}}^2\rho_{sat,l}{a_{sat,l}}^2(\rho_{sat,v} - \rho_{sat,l})}{{\rho_{sat,v}}^2{a_{sat,v}}^2 - {\rho_{sat,l}}^2{a_{sat,l}}^2}$$

Equation of State — ISM Model

- ISM Model (Ihm, Song, Mason)

$$\frac{P}{\rho kT} = 1 + \frac{(B-\alpha)\rho}{1+0.22\lambda b\rho} + \frac{\alpha\rho}{1-\lambda b\rho} \Rightarrow G(b\rho)^{-1} = \alpha\rho\left[Z - 1 + \frac{(\alpha - B)\rho}{1+0.22\lambda b\rho}\right]^{-1} = 1 - \lambda b\rho$$

Where

$$B\rho_m = 0.403891 - 0.076484(\Delta H_{vap}/RT)^2 - 0.0002504(\Delta H_{vap}/RT)^4$$

$$\alpha\rho_m = a_1 \exp[-c_1(RT/\Delta H_{vap})] + a_2\{1 - \exp[-c_2(\Delta H_{vap}/RT)^{1/4}]\}$$

$$b\rho_m = a_1[1 - c_1(RT/\Delta H_{vap})]\exp[-c_1(RT/\Delta H_{vap})]$$
$$+ a_2\{1 - [1 + 1/4c_2(\Delta H_{vap}/RT)^{1/4}]\exp[-c_2(\Delta H_{vap}/RT)^{1/4}]\}$$

$$a_1 = -0.1053, \qquad a_2 = 2.9359$$

$$c_1 = 5.7862, \qquad c_2 = 0.7966$$

Predict the density of Hg from the melting point up to 100 degree above the boiling Temperature.

Mathematical Model for Hg Flow — Equation of State(4)

- SESAME Library

  The SESAME opacity tables are compatible with the equation of state SESAME tables

- Summation Method (Ahmed Hassanein)

  Sum of cold compression, ion thermal, and electron thermal terms

$$E = E_c + E_I + E_e$$
$$P = P_c + P_I + P_e$$

Where

E_c(Energy of cold compression): $E_c = E_{c0}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right]\frac{\rho}{\rho_0} + P_{c0}(1-\frac{\rho}{\rho_0})$; $E_{c0} = \frac{P_{c0}}{\gamma-1}$;

P_c(Pressure of cold compression): $P_c = P_{c0}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right]$; $P_{c0} = \frac{\rho c_s^2}{\gamma}$;

E_I(Ion thermal energy): $E_I = 3nk(T - T_0)$;

P_I(Ion thermal pressure): $P_I = G_I E_I$;

E_e(Electron thermal energy): $E_e = \frac{1}{2}\beta(T-T_0)^2$, $\beta = \beta_0(\frac{\rho_0}{\rho})^{2/3}$;

P_e(Electron thermal pressure): $P_e = G_e E_e$;

G_I, G_e: the Gruneisen coefficients for the ion and electron;

C_s: the speed of sound in the liquid target;