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Magnetohydrodynamics of a Continuous Mercury Jet Coaxially Entering a Solenoid

1 Introduction

We consider a continuous cylindrical jet of mercury of constant mass density $\rho = 13.6$ g/cm^2 and conductivity $\sigma \approx 10^{16}$/s (Gaussian units) that enters a semi-infinite solenoid magnet of peak axial field $B_0 = 20$ T and diameter $D = 20$ cm. The jet has radius $R(z) \approx 1$ cm, and its velocity components will be labeled $v_r(r,z)$ and $v_z(r,z)$ in a cylindrical coordinate system in which the jet axis, and the magnetic field axis, is the z axis. The solenoid coil extends from $z = 0$ to $= \infty$. The initial velocity $v_z(r, -\infty)$ of the jet is of order 20 m/s.

The magnetic diffusion time $\tau = 4\pi\sigma R^2/c^2$ is small compared to the time scale D/v_z over which the magnetic field changes in the rest frame of the jet, so the magnetic Reynolds number $\mathcal{R} = v_z\tau/D$ is much less than unity. As a consequence, the induced magnetic field is small compared to that of the solenoid, which is "fully diffused" into the mercury. We pursue a solution in which we ignore the induced magnetic field, and calculate the induced electric field in the local rest frame of the jet via Faraday's law. The fluid flow velocity $\mathbf{v}(r,z)$ is assumed to be incompressible ($\nabla \cdot \mathbf{v} = 0$), and is expanded in a power series in r, with coefficients being functions of z, through second order.

2 Equation of Motion of the Fluid

The equation of motion of the mercury is [1]

$$\rho\frac{d\mathbf{v}}{dt} = \rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \rho_{\rm charge}\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B} + \eta[\nabla^2\mathbf{v} + \nabla(\nabla\cdot\mathbf{v})] + \rho\mathbf{g}, \qquad (1)$$

where P is the pressure, $\rho_{\rm charge}$ is the electric charge density, $\mathbf{E}$ is the electric field, $\mathbf{j}$ is the current density, c is the speed of light, $\mathbf{B}$ is the magnetic field, $\eta = 0.0015$ g/(s-cm) is the viscosity and $\mathbf{g}$ is the acceleration due to gravity. At the free surface of the mercury jet, the surface tension, $\gamma = 470$ dyne/cm, plays a role discussed later.

In this note we ignore gravity and viscosity, and seek only a steady-state solution in which $\partial\mathbf{v}/\partial t = 0$ (although we make some remarks about transient magnetic effects in sec. 5). We also assume that mercury is incompressible, so that

$$\nabla\cdot\mathbf{v} = 0. \qquad (2)$$

A well-known consequence of incompressibility is that $\langle v\rangle A$ is constant, where $\langle v\rangle$ is the average velocity along the jet axis and A is the cross sectional area of the jet.

Then, the equation of motion (1) reduces to

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \rho_{\text{charge}}\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B}. \tag{3}$$

3 Maxwell's Equations

While this note will emphasize steady-state solutions, we have a larger interest in a pulsed jet, so we briefly discuss the possible time dependence of the electromagnetic fields. The Maxwell equations are

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 4\pi\rho_{\text{charge}} \approx 0, && (4)\\
\nabla\cdot\mathbf{B} &= 0, && (5)\\
\nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, && (6)\\
\nabla\times\mathbf{B} &\approx \frac{4\pi}{c}\mathbf{j}, && (7)
\end{aligned}$$

where we ignore charge separation in the mercury, and also ignore the displacement current [3], *i.e.*, we ignore high-frequency phenomena such as plasma oscillations, and we take the permeability μ of mercury to be unity.

In this approximation, the divergence of eq. (7) yields

$$\nabla\cdot\mathbf{j} = 0, \tag{8}$$

and hence ρ_{charge} is constant in time.

In the present problem, the magnetic field of the solenoid can be considered as a known external, time-independent field $\mathbf{B}_{\text{ext}}$ that obeys $\nabla\times\mathbf{B}_{\text{ext}} = 0$ and $\nabla^2\mathbf{B}_{\text{ext}} = 0$ in the region of the mercury jet. The equations (4)-(7) then apply to the induced electric and magnetic fields $\mathbf{E}_{\text{ind}}$ and $\mathbf{B}_{\text{ind}}$.

Because the conductivity of mercury is not large, it may be that $\mathbf{B}_{\text{ind}} \ll \mathbf{B}_{\text{ext}}$ and an analysis of the motion of the jet can be carried out while ignoring $\mathbf{B}_{\text{ind}}$.

4 Ohm's Law

The current density is related by Ohm's law in the local rest frame of the mercury,

$$\mathbf{j}^\star = \sigma\mathbf{E}^\star. \tag{9}$$

Since $v \ll c$, rest frame quantities are related their lab frame values by

$$\mathbf{j}^\star \approx \mathbf{j} - \rho_{\text{charge}}\mathbf{v} \approx \mathbf{j}, \tag{10}$$

and

$$\mathbf{E}^\star \approx \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}, \tag{11}$$

ignoring terms of order $v^2/c^2 \ll 1$. We can combine eqs. (9) and (11) to obtain

$$\mathbf{j} = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \tag{12}$$

or equivalently,

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \frac{\mathbf{v}}{c} \times \mathbf{B}. \tag{13}$$

Equation (12) suggests that there will be a current flow due the the Lorentz force of the external magnetic field on the metal jet. In the first approximation, this current would be transverse to the jet velocity, and hence transverse to the jet axis. Such a current quickly ($\tau \approx 4\pi/\sigma \approx 10^{-15}$ s) leads to the accumulation of charge on the surface of the jet, which sets up an electric field that cancels the transverse component of the $\mathbf{v} \times \mathbf{B}_{\rm ext}$ force, and hence cancels the transverse current.

The only possibility for steady currents in an axisymmetry jet in an axisymmetric magnetic field is for azimuthal currents. In this case, no electric field due to charge separation exists to cancel the currents. The azimuthal currents are considered in detail in sec. 6.4.

5 Magnetic Diffusion Time and Reynolds Number

In a medium with high enough conductivity, any external magnetic field is cancelled its interior by the field that arises from induced surface currents. We argue that mercury is not a good conductor in this sense, and that an external magnetic field penetrates into the interior of mercury essentially without cancellation.

Together with eq. (13), the third and fourth Maxwell equations, (6) and (7), yield

$$\frac{\partial \mathbf{B}_{\rm ind}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_{\rm ind}) + \frac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B}_{\rm ind}. \tag{14}$$

For low fluid velocities, eq. (14) has the form of a diffusion equation. In our example of jet of radius $R \approx 1$ cm, the maximum of $\nabla^2 \mathbf{B}_{\rm ind}$ is approximately $\mathbf{B}_{\rm ind}/R^2$, so the induced magnetic field induced as the jet first enters the magnet dies out with characteristic time

$$\tau \approx \frac{4\pi\sigma R^2}{c^2} \approx \frac{4\pi \cdot 10^{16} \cdot (1)^2}{(3 \times 10^{10})^2} \approx 10^{-4} \text{ s}, \tag{15}$$

The spatial scale for variation of the solenoid field is its diameter D, so the time scale for changes in the motion of the jet is of order $D/v_z \approx 0.01$ s $\approx 100\tau$. Thus, transient magnetic effects die away much more rapidly than changes in the motion of the jet, and the induced magnetic field never grows large enough to cancel the external field. Hence, the magnetic field inside the mercury can be well approximated as that of the external solenoid [2]. This result is further validated in sec. 6.3.

The so-called magnetic Reynolds number is

$$\mathcal{R}_M = \frac{v_z \tau}{D} \approx 0.01, \tag{16}$$

whose small value is a reminder that the magnetic field lines are "fully diffused" into the mercury (and NOT "frozen in").

6 Approximate Equation of Motion

6.1 Power Series Expansion for the Velocity Field of a Circularly Symmetric Axial Jet

We analyze a circular jet with zero initial angular momentum about its axis as it moves coaxially through a circularly symmetry magnetic field. Then, the jet has zero angular momentum along its entire trajectory, and its azimuthal velocity vanishes,

$$v_\phi = 0. \tag{17}$$

With condition (17), this implies that

$$\frac{1}{r}\frac{\partial r v_r}{\partial r} = -\frac{\partial v_z}{\partial z}. \tag{18}$$

We will make a series expansion of v_r and v_z of the form

$$v_z(r,z) = \sum_n f_n(z) r^n, \tag{19}$$

$$v_r(r,z) = \sum_n g_n(z) r^{n+1}, \tag{20}$$

noting that cylindrical symmetry requires $v_r(0,z) = 0$. The divergence condition (2) tells us that $g_n = -f_n'/(n+2)$, where the $'$ means d/dz, so the the radial velocity expansion is

$$v_r(r,z) = -\sum_n \frac{f_n'(z)}{n+2} r^{n+1}, \tag{21}$$

If v_z decreases, then v_r increases according to eq. (21) and the radius of the jet grows, consistent with incompressible flow in which $\langle v_z \rangle A = \text{constant}$.

It is tempting at this stage to suppose also that the motion of the jet is irrotational, *i.e.*, that

$$\nabla \times \mathbf{v} = 0. \tag{22}$$

However, a simple model of eddy current effects in the liquid metal jet [2] suggests that a velocity shear will result as the jet enters the magnetic field, leading to nonzero curl of the velocity.

Our plan is to use the expansions (19) and (21) for the velocity components in the equation of motion (3), keeping terms in f_0, f_1 and f_2. Then,

$$v_z \approx f_0 + r f_1 + r^2 f_2, \tag{23}$$

$$v_r \approx -\frac{r}{2} f_0' - \frac{r^2}{3} f_1' - \frac{r^3}{4} f_2'. \tag{24}$$

We will find that since $f_1(-\infty) = 0 = f_1'(-\infty)$, f_1 vanishes everywhere. Also, to deduce a differential equation for f_2'' we will have to keep terms up to r^3 in the radial equations of motion and up to r^4 in the axial equation. To these orders we find that

$$\begin{aligned}
\rho(\mathbf{v}\cdot\nabla)v_r \quad &\approx \quad \frac{\rho r}{4}[(f_0')^2 - 2f_0 f_0''] \\
&+ \frac{\rho r^2}{6}[3f_0'f_1' - 3f_1 f_0'' - 2f_0 f_1''] \\
&+ \frac{\rho r^3}{36}[18f_0'f_2' + 8(f_1')^2 - 9f_0 f_2'' - 12f_1 f_1'' - 18f_2 f_0''], \qquad (25) \\
\rho(\mathbf{v}\cdot\nabla)v_z \quad &\approx \quad \rho f_0 f_0' \\
&+ \frac{\rho r}{2}[f_1 f_0' + 2f_0 f_1'] \\
&+ \frac{\rho r^2}{3}[2f_1 f_1' + 3f_0 f_2'] \\
&+ \frac{\rho r^3}{12}[4f_2 f_1' + 9f_1 f_2'] \\
&+ \frac{\rho r^4}{2} f_2 f_2'. \qquad (26)
\end{aligned}$$

6.2 Power Series Expansion of the Pressure

We expand the pressure as

$$P(r,z) \approx \rho[q_0(z) + q_1(z)r + q_2(z)r^2 + q_3(z)r^3 + q_4(z)r^4]. \qquad (27)$$

This is subject to the condition that the external pressure vanishes at the free surface at radius $R(z)$, but the surface tension γ provides a small nonzero pressure γ/R, so that

$$\frac{\gamma\rho}{R} = q_0 + q_1 R + q_2 R^2 + q_3 R^3 + q_4 R^4. \qquad (28)$$

All the q_i except q_0 and all derivatives q_i' vanish at $z = -\infty$. We will find that both q_1 and q_3 are zero.

6.3 Power Series Expansion of the Magnetic Field of the Solenoid

The cylindrically symmetric magnetic field of the solenoid obeys $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{B} = 0$, and so can be expanded in terms of the axial field as

$$B_z(r,z) = \sum_n (-1)^n \frac{B^{(2n)}(z)}{(n!)^2}\left(\frac{r}{2}\right)^{2n} = B(z) - \frac{B''(z)r^2}{4} + ..., \qquad (29)$$

and

$$B_r(r,z) = \sum_n (-1)^{n+1} \frac{B^{(2n+1)}(z)}{(n+1)(n!)^2}\left(\frac{r}{2}\right)^{2n+1} = -\frac{B'(z)r}{2} + \frac{B'''(z)r^3}{16} - ..., \qquad (30)$$

where

$$B(z) \equiv B_z(0,z), \qquad B^{(n)} = \frac{d^n B}{dz^n}, \qquad \text{and} \qquad B' = B^{(1)},\ etc. \qquad (31)$$

6.4 Power Series Expansion of Current Density and of the Lorentz Force

We desire an analysis of the motion of the mercury that proceeds without calculation the small induced electric and magnetic fields. In this case, we cannot use the 4th Maxwell equation (7) to replace the current density in eq. (3) by the curl of the magnetic field (which vanishes in the proposed approximation).

Rather, we deduce the form of the current density via Ohm's law (12). The lab frame electric field is zero in the present case, so we have

$$\mathbf{j} = \frac{\sigma}{c}\mathbf{v} \times \mathbf{B}. \tag{32}$$

Both $\mathbf{v}$ and $\mathbf{B}$ are assumed to be axisymmetric, so (as discussed in sec. 4) the only nonvanishing component of the current density is

$$j_\varphi = \frac{\sigma}{c}(v_z B_r - v_r B_z) \approx \frac{\sigma}{c}\left(-\frac{r v_z B'}{2} + \frac{r^3 v_z B'''}{16} - v_r B + \frac{r^2 v_r B''}{4}\right), \tag{33}$$

using the expansions (29)-(30), which presumes that we can neglect the magnetic fields induced by the eddy currents.

The nonzero components of the $\mathbf{j} \times \mathbf{B}$ force are then

$$\begin{aligned}
\left(\frac{\mathbf{j}}{c} \times \mathbf{B}\right)_r &= \frac{j_\varphi B_z}{c} \\
&\approx \frac{\sigma}{c^2}\left(-\frac{r v_z B B'}{2} + \frac{r^3 v_z B' B''}{8} + \frac{r^3 v_z B B'''}{16} - v_r B^2 + \frac{r^2 v_r B B''}{2}\right), \qquad (34) \\
\left(\frac{\mathbf{j}}{c} \times \mathbf{B}\right)_z &= -\frac{j_\varphi B_r}{c} \\
&\approx \frac{\sigma}{c^2}\left(-\frac{r^2 v_z (B')^2}{4} + \frac{r^4 v_z B' B'''}{16} - \frac{r v_r B B'}{2} + \frac{r^3 v_r B' B''}{8} + \frac{r^3 v_r B B'''}{16}\right). \qquad (35)
\end{aligned}$$

The lowest order term in the $\mathbf{j} \times \mathbf{B}$ force for an axial jet in a solenoid is the first term in eq. (34), the so-called radial pinch. This contributes directly to the creation of a nonzero radial velocity v_r, and also leads to an internal pressure in the jet whose gradient affects both v_r and v_z.

The leading axial force term in eq. (35) is the first, which reduces the axial velocity wherever the axial magnetic field is varying in space.

By Lenz' law, we expect the higher order terms in the $\mathbf{j} \times \mathbf{B}$ force to oppose the lowest order effect, and damp the perturbations due to the magnetic field. Thus, if the radial pinch produced a negative radial velocity, the third term in eq. (35) would increase the axial velocity (as the jet enters the solenoid), in contrast to the first term. However, the condition of incompressibility, $\langle v_z \rangle A =$ constant, implies that as v_z decreases on entering the magnet, v_r must grow so that A grows despite the radial pinch. As a consequence, the third term in eq. (35) will also cause a reduction in v_z that appears to be important.

Even ignoring the higher-order terms in eq. (33) for the current density, the induced magnetic field is predicted to be small, as claimed in sec. 5. To see this, we use the 4th Maxwell equation,

$$(\nabla \times \mathbf{B}_{\rm ind})_\varphi \approx -\frac{\partial B_{z,\rm ind}}{\partial r} \approx \frac{4\pi}{c} j_\varphi \approx -\frac{2\pi\sigma r v_z}{c^2}\frac{\partial B_{z,\rm ext}}{\partial z}, \tag{36}$$

which integrates to

$$B_{z,\rm ind} \approx \frac{\pi\sigma r^2 v_z}{c^2}\frac{\partial B_{z,\rm ext}}{\partial z} \lesssim \frac{\pi\sigma R^2 v_z}{c^2}\frac{B_0}{D} \approx \tau\frac{v_z}{D}B_0 \approx \mathcal{R}_M B_0 \approx 0.01 B_0. \tag{37}$$

Another way to deduce eq. (33) is via $\mathbf{j} \approx \mathbf{j}^\star = \sigma\mathbf{E}^\star$, in terms of the electric field in the the local rest frame of the fluid. While the field $\mathbf{B}$ of the solenoid in the lab frame is time independent in our approximation, the corresponding field $\mathbf{B}^\star$ is time dependent, and induces the field $\mathbf{E}^\star$ according to Faraday's law,

$$\oint \mathbf{E}^\star \cdot d\mathbf{l}^\star = -\frac{1}{c}\frac{d\Phi^\star}{dt^\star}. \tag{38}$$

However, we can take value of the the magnetic field $\mathbf{B}^\star$ to be the same as that in the lab frame, since the induced lab-frame electric field will be proportional to v/c and the resulting correction to $\mathbf{B}^\star$ will be of order $v^2/c^2 \ll 1$. Likewise, $t^\star = t$ plus corrections of order $v^2/c^2 \ll 1$.

We analyze a ring of radius $r = r^\star$, for which the lefthand side of eq. (38) is $2\pi r E^\star_\varphi$. The magnetic flux through this ring varies with time in the $\star$ frame because solenoid appears to be moving with respect to the ring, and because the radius of the ring is changing at rate $v_r^\star \approx v_r$. We can calculate $d\Phi^\star/dt$ using lab-frame quantities via the convective derivative, $d/dt^\star = \partial/\partial t + (\mathbf{v}\cdot\nabla)$. Thus,

$$j_\varphi \approx \sigma E^\star_\varphi \approx -\frac{\sigma}{2\pi r c}(\mathbf{v}\cdot\nabla)\Phi = -\frac{\sigma}{2\pi r c}\left(v_z\frac{\partial\Phi}{\partial z} + v_r\frac{\partial\Phi}{\partial r}\right). \tag{39}$$

Equation (33) follows using the magnetic flux through the ring,

$$\Phi = \int_0^r B_z(r)\, 2\pi r dr \approx \pi r^2 B - \frac{\pi r^4 B''}{8}, \tag{40}$$

in the approximation of eq. (29).

6.5 Power Series Expansion of the Equations of Motion

With eqs. (23)-(24), (27) and (34)-(35), the righthand sides of the equations of motion (3) are, to order r^3 in the radial equation and order r^4 in the axial equation,

$$\begin{aligned}\rho(\mathbf{v}\cdot\nabla)v_r \;\approx\; & -\rho q_1 \\ & -2\rho r q_2 + \frac{\sigma r}{2c^2}[f_0' B^2 - f_0 B B'] \\ & -3\rho r^2 q_3 + \frac{\sigma r^2}{6c^2}[2 f_1' B^2 - 3 f_1 B B']\end{aligned}$$

$$
\begin{aligned}
& - 4\rho r^3 q_4 + \frac{\sigma r^3}{16c^2}[4f_2' B^2 - 8f_2 BB' - 4f_0' BB'' + f_0(BB''' + 2B'B'')], \quad (41) \\
\rho(\mathbf{v}\cdot\nabla)v_z \approx & - \rho q_0' \\
& - \rho r q_1' \\
& - \rho r^2 q_2' + \frac{\sigma r^2}{4c^2}[f_0' BB' - f_0(B')^2] \\
& - \rho r^3 q_3' + \frac{\sigma r^3}{12c^2}[2f_1' BB' - 3f_1(B')^2] \\
& - \rho r^4 q_4' + \frac{\sigma r^4}{32c^2}[4f_2' BB' - 8f_2(B')^2 + 2f_0 B'B''' - f_0'(BB''' + 2B'B'')] \quad (42)
\end{aligned}
$$

We now compare eqs. (25)-(26) and (41)-(42) order by order to obtain 9 equations in the 9 unknown functions f_0, f_1, f_2, q_0, q_1, q_2, q_3, q_4 and R.

$$
\begin{aligned}
0 &= q_1, & (43)\\
(f_0')^2 - 2f_0 f_0'' &= -8q_2 + \frac{2\sigma}{\rho c^2}[f_0' B^2 - f_0 BB'], & (44)\\
3f_0' f_1' - 3f_1 f_0'' - 2f_0 f_1'' &= -18q_3 + \frac{\sigma}{\rho c^2}[2f_1' B^2 - 3f_1 BB'], & (45)\\
18f_0' f_2' + 8(f_1')^2 - 9f_0 f_2'' - 12f_1 f_1'' - 18f_2 f_0'' &= -144q_4 + \frac{9\sigma}{4\rho c^2}[4f_2' B^2 - 8f_2 BB' - 4f_0' BB'' & \\
&\qquad + f_0(BB''' + 2B'B'')], & (46)\\
(f_0^2)' &= -2q_0', & (47)\\
f_1 f_0' + 2f_0 f_1' &= -2q_1', & (48)\\
2f_1 f_1' + 3f_0 f_2' &= -3q_2' + \frac{3\sigma}{4\rho c^2}[f_0' BB' - f_0(B')^2], & (49)\\
4f_2 f_1' + 9f_1 f_2' &= -12q_3' + \frac{\sigma}{\rho c^2}[2f_1' BB' - 3f_1(B')^2], & (50)\\
f_2 f_2' &= -2q_4' + \frac{\sigma r^4}{16\rho c^2}[4f_2' BB' - 8f_2(B')^2 + 2f_0 B'B''' & \\
&\qquad - f_0'(BB''' + 2B'B'')]. & (51)
\end{aligned}
$$

Not only is $q_1 = 0$ according to eq. (43), but f_1 and q_3 vanish as well since eqs. (45), (48) and (50) imply that if f_1, f_1', q_3 and q_3' all vanish at some value of z, then they vanish at all z; and our initial condition is that all of these vanish at $z = -\infty$.

There are now 6 unknown functions, f_0, f_2, q_0, q_2, q_4 and R, but only 5 of the 9 equations of motion (43)-(51) remain. In particular, we need a relation for $R(z)$.

A simple and numerically stable relation for the jet radius R follows from the assumption of incompressibility. Namely, the flux of liquid is constant across any plane of constant z:

$$
\Phi_z = \int_0^R v_z(r)\, 2\pi r dr \approx \pi R^2 f_0 + \frac{\pi R^4 f_2}{2} = \pi R^2(-\infty) v_z(-\infty), \quad (52)
$$

using eq. (23). Then,

$$
R^2(z) = \frac{-f_0 + \sqrt{f_0^2 + 2R^2(-\infty) v_z(-\infty) f_2}}{f_2}. \quad (53)
$$

We integrate eq. (47) to find

$$q_0 = \frac{\gamma}{\rho R(-\infty)} + \frac{1}{2}v_z^2(-\infty) - \frac{1}{2}f_0^2, \tag{54}$$

noting that the pressure in the jet is only γ/R due to surface tension at $z = -\infty$. Since $q_0 = P(0,z)/\rho$, we recognize eq. (54) as Bernoulli's equation, $P + \rho v^2/2 =$ constant, along the axis of the jet. This equation is formally consistent with negative pressure, and implies that the axial velocity will be high if the axial pressure becomes negative.

We do not find a version of Bernoulli's equation at nonzero radius. Since Joule heating due to eddy currents occurs at any nonzero radius, we do not expect Bernoulli's equation, which represents conservation of mechanical energy, to apply there.

The remaining unknown functions, f_0, f_2 q_2 and q_4, will be obtained by numerical integration. Function f_0 will be determined from eq. (44) according to

$$f_0'' \approx \frac{(f_0')^2}{2f_0} + \frac{4q_2}{f_0} + \frac{\sigma}{\rho c^2}\left(BB' - \frac{f_0' B^2}{f_0}\right). \tag{55}$$

Equation (46) determines f_2'' via

$$f_2'' \approx \frac{2f_0' f_2'}{f_0} - \frac{2f_2 f_0''}{f_0} + \frac{16q_4}{f_0} - \frac{\sigma}{4\rho c^2}\left(4\frac{f_2' B^2}{f_0} - 8\frac{f_2 BB'}{f_0} - 4\frac{f_0' BB''}{f_0} + BB''' + 2B'B''\right). \tag{56}$$

Equation (49) specifies q_2' as

$$q_2' \approx -f_0 f_2' + \frac{\sigma}{4\rho c^2}[f_0' BB' - f_0(B')^2], \tag{57}$$

and eq. (51) specifies q_4' as

$$q_4' \approx -\frac{f_2 f_2'}{2} + \frac{\sigma}{32\rho c^2}[4f_2' BB' - 8f_2(B')^2 + 2f_0 B'B'' - f_0'(BB''' + 2B'B'')]. \tag{58}$$

We have the option to determine q_4 via the pressure boundary condition (28), which now tells us that

$$q_4 R^4 + q_2 R^2 + q_0 = \frac{\gamma}{\rho R}, \tag{59}$$

and so

$$q_4 = \frac{\gamma}{\rho R^5} - \frac{q_2}{R^2} - \frac{q_0}{R^4}. \tag{60}$$

However, this does not appear to stabilize the numerical integration; it seems better to use eq. (58).

7 Motion in a Semi-Infinite Solenoid

For a physical solenoid of central field B_0 and diameter D, the corresponding axial magnetic field of the semi-infinite solenoid is

$$B_z(0,z) = \frac{B_0}{2}\left(1 + \frac{z}{\sqrt{(D/2)^2 + z^2}}\right), \tag{61}$$

whose derivatives are

$$B_z' = \frac{dB_z(0,z)}{dz} = \frac{B_0}{2}\frac{(D/2)^2}{[(D/2)^2+z^2]^{3/2}}, \tag{62}$$

$$B_z'' = \frac{d^2B_z(0,z)}{dz^2} = -\frac{3B_0(D/2)^2}{2}\frac{z}{[(D/2)^2+z^2]^{5/2}}, \tag{63}$$

$$B_z''' = \frac{d^3B_z(0,z)}{dz^3} = -\frac{3B_0(D/2)^2}{2}\frac{(D/2)^2-4z^2}{[(D/2)^2+z^2]^{7/2}}. \tag{64}$$

As of 12/9/00, the numerical integration of eqs. (55)-(58) appears to imply that the jet comes to rest at $z \approx -D$. That is, it can't enter the magnet...

8 References

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