Geometry of Viewing Mercury Drops

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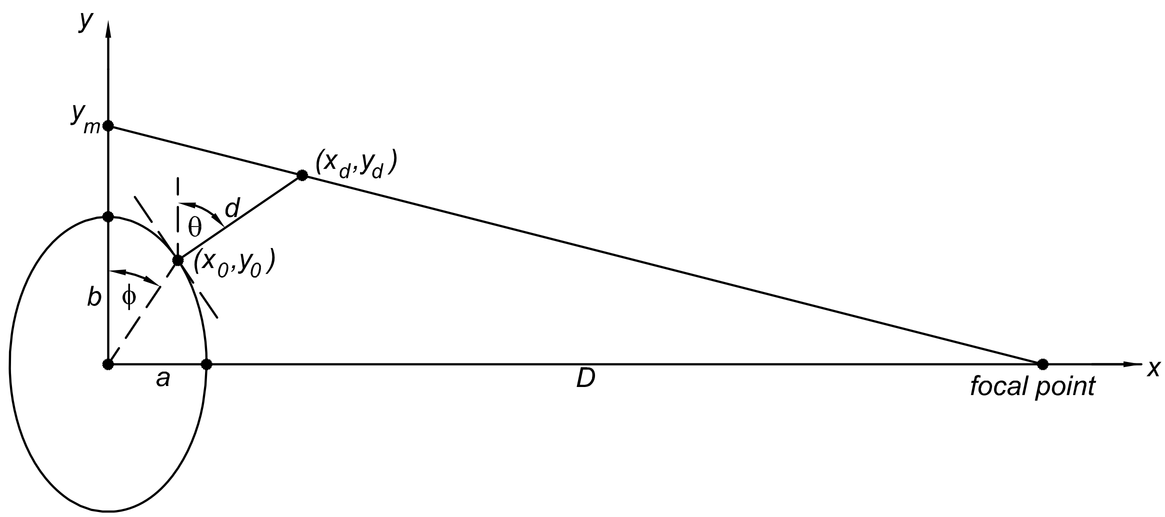
(December 3, 2008)

In the MERIT experiment, the mercury jet, and any droplets ejected from it by the proton beam interaction, were viewed via shadow photography from a distance $D = 9.15\,\mathrm{cm}$ from the center of the jet.



The jet may have had an elliptical cross section, so we describe the surface of the jet by the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

Of course, if the jet were circular with radius a, then $b = a$.

Can we get any indication of whether the jet were circular or elliptical using only our shadow photography measurements?

These measurements describe the projection $y_m(t)$ onto the y (vertical) axis of a ray from the observer that passes through a droplet at position $(x_d(t), y_d(t))$,

An interesting question is whether the droplets leave the surface of the jet in a direction perpendicular to the surface, or at some other angle. Here, I assume that they leave perpendicularly, as shown above.

Suppose a droplet leaves the surface with velocity v_0 at time t_0 from point (x_0, y_0). Then, at time $t > t_0$, it has traveled distance

$$d = v_0(t - t_0), \tag{2}$$

assuming that the velocity stays constant. The position of the drop is

$$x_d = x_0 + d\sin\theta, \qquad y_d = y_0 + d\cos\theta. \tag{3}$$

The surface of the elliptical jet is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{or} \qquad x = \frac{a}{b}\sqrt{b^2 - y^2}, \qquad \text{or} \qquad y = \frac{b}{a}\sqrt{a^2 - x^2}. \tag{4}$$

The tangent to the surface of the jet at the point (x_0, y_0) obeys

$$\text{slope} = \frac{dx}{dy} = -\frac{a}{b}\frac{y_0}{x_0}\,. \tag{5}$$

The droplet moves perpendicular to the slope, so

$$\tan\theta = \frac{-1}{\text{slope}} = -\frac{dy}{dx} = \frac{b}{a}\frac{x_0}{y_0}\,, \qquad \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{x_0}{a}\,, \qquad \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{y_0}{b}\,. \tag{6}$$

Using these in eq. (3), we have

$$x_d = (a+d)\sin\theta, \qquad y_d = (b+d)\cos\theta, \tag{7}$$

When the drop is viewed via shadow photography from a point at distance D from the center of the jet, the position of the drop, y_m, as projected onto the y axis is

$$\begin{aligned} y_m &= y_d\frac{D}{D-x_d} + c \approx y_d\left(1+\frac{x_d}{D}\right) + c \\ &= [b\cos\theta + c] + v_0(t-t_0)\cos\theta + \frac{[a+v_0(t-t_0)][b+v_0(t-t_0)]}{2D}\sin 2\theta, \end{aligned} \tag{8}$$

where c is the y-coordinate of the center of the jet.

The apparent velocity of the droplet along the y axis is

$$v_m = \frac{dy_m}{dt} \approx v_0\left[\cos\theta + \frac{a+b+2v_0(t-t_0)}{2D}\sin 2\theta\right]. \tag{9}$$

If the droplet is moving towards the observer, $0 < \theta < 180°$, then the apparent velocity v_m increasing slightly with time (if air resistance can be ignored). Do we have any evidence in our data for this small effect?

The earliest time t_{0m} that a droplet can be seen via shadow photography is when $y_m \approx b$,[1] so that

$$t_{0m} \approx t_0 + \frac{b(1-\cos\theta)}{v_0\cos\theta} \approx t_0 + \frac{b(1-v_m/v_0)}{v_m}\,, \tag{10}$$

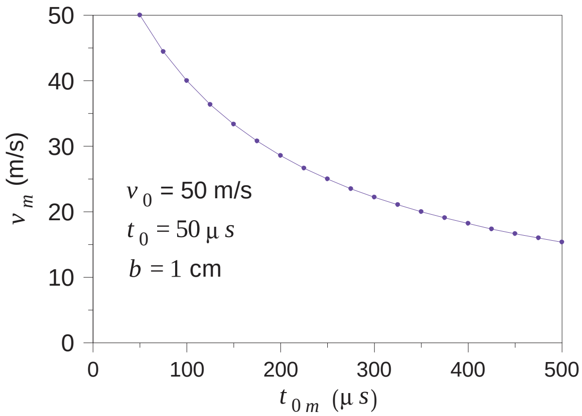
and

$$v_m \approx \frac{v_0}{1+v_0(t_{0m}-t_0)/b}\,. \tag{11}$$

In the first approximation, v_m depends on t_{0m} only through the height b of the jet.

An example of the relation between v_m and t_{0m} is shown in the figure on the next page.

[1] In case of a circular jet, the minimum measureable y_m is $b/\sqrt{1-b^2/D^2} \approx b(1+b^2/2D^2)$. We ignore the second-order correction.



We now face a statistical question: Are the droplets distributed uniformly in angle θ, or perhaps uniformly in angle ϕ, or perhaps they are equally probable to be emitted from any point on the surface?

1. Uniform in θ.

$$P(\theta)\, d\theta = \frac{d\theta}{2\pi}\,. \tag{12}$$

2. Uniform in ϕ.

$$\tan\phi = \frac{x}{y}\,, \qquad \sin\phi = \frac{x}{\sqrt{x^2+y^2}}\,, \qquad \cos\phi = \frac{y}{\sqrt{x^2+y^2}}\,, \tag{13}$$

and from eq. (6),

$$x = a\sin\theta, \qquad y = b\cos\theta. \tag{14}$$

After some algebra,

$$d\phi = \frac{ab}{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta. \tag{15}$$

Hence,

$$P(\theta)\, d\theta = P(\phi)\, d\phi = \frac{d\phi}{2\pi} = \frac{ab}{a^2\sin^2\theta + b^2\cos^2\theta}\frac{d\theta}{2\pi}\,. \tag{16}$$

3. Uniform in position s around the circumference C of the ellipse.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta. \tag{17}$$

Hence,

$$P(\theta)\, d\theta = P(s)\, ds = \frac{ds}{C} \approx \frac{2\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{3(a+b) - \sqrt{(3a+b)(a+3b)}}\frac{d\theta}{2\pi}\,, \tag{18}$$

using Ramanujan's (very good!) approximation for the circumference of an ellipse.