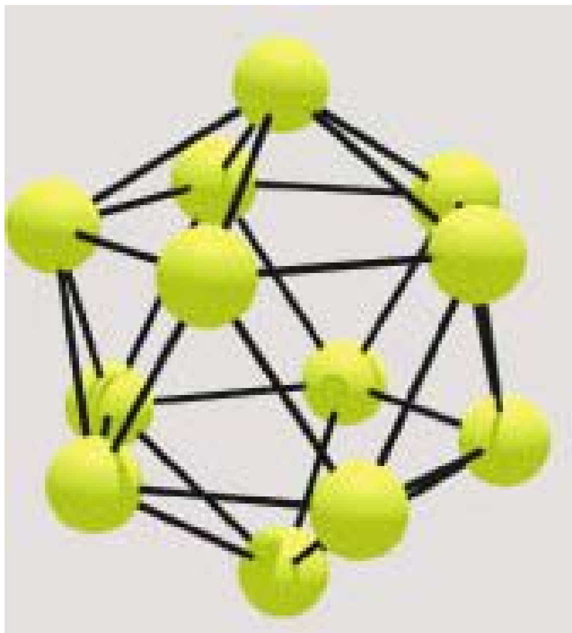
Review of Analytic Models of the Magnetohydrodynamics of Liquid Metal Jets



[Pb_{12} “molecule” in liquid lead, H. Reichert *et al.*, Nature **408**, 839 (2000)]

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Feasibility Study 2 Workshop, BNL

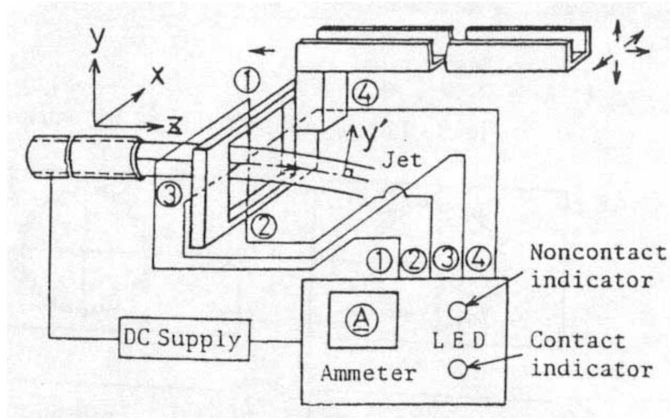
http://puhep1.princeton.edu/mumu/target/

Some History

- 1987: S. Oshima: quadrupole distortion of a mercury jet.
- 1988: C. Johnson: mercury jet as a possible proton target.
- 1995: R. Palmer, R. Weggel: muon collider primary target inside a 20-T solenoid. Mercury target a possibility.
- 1997: K. McDonald, R. Palmer, R. Weggel: eddy current effects in liquid jets modeled using rings.
- 6/2000: R. Samulyak: begins full simulation of jet magnetohydrodynamics.
- 10/2000: V. Lebedev: challenges us to make analytic approximations to the full magnetohydrodynamic equations.
- 10/2000: P. Thieberger: adds pressure-gradient effects.
- 10/2000: R. Palmer, S. Kahn: filament approximation to jet, including pressure gradients and quadrupole distortion.
- 11/2000: K. McDonald: attempt to approximate the full magnetohydrodynamic equations to 2nd order.

Quadrupole Distortion of Jet When B ⊥ v

S. Oshima *et al.*,
JSME Int. J. **30**, 437 (1987).
2-T **B** ⊥ to **v**.



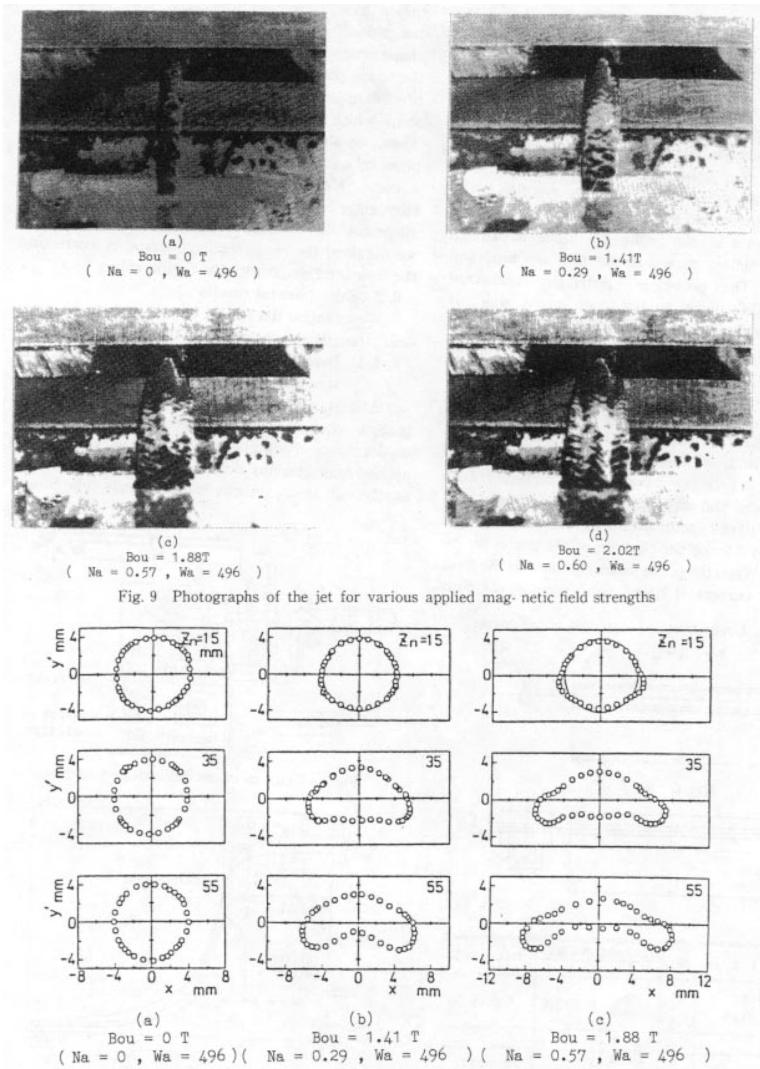


Fig. 9 Photographs of the jet for various applied mag- netic field strengths

Fig. 10 Cross-sectional shape of the jet obtained by spot a electrode probe

Analytic calculation agrees with the data.

Magnetohydrodynamics of a Liquid Jet

$$\rho \frac{d\mathbf{v}}{dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \rho_{\text{charge}} \mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B} + \eta[\nabla^2 \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})] + \rho \mathbf{g},$$

$\rho = 13.6\ \text{g/cm}^2$ = density of mercury,

$\mathbf{v}$ = velocity of jet,

P = pressure,

$\rho_{\text{charge}} \approx 0$ = electric charge density. But surface charge $\neq 0$.

$\mathbf{J}$ = current density,

$\eta = 0.0015\ \text{g/(s-cm)}$ = viscosity of mercury,

g = acceleration due to gravity,

At the free surface, $\gamma = 470$ dyne/cm = surface tension plays a role.

Ignore viscosity and gravity. Consider steady, incompressible flow:

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \frac{\mathbf{J}}{c} \times \mathbf{B}.$$

This equation is **nonlinear**.

Linearized Equation of Motion

IF $\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$ where $|\delta\mathbf{v}| \ll |\mathbf{v}_0|$, then $(\mathbf{v}\cdot\nabla)\mathbf{v} \approx (\mathbf{v}_0\cdot\nabla)\delta\mathbf{v}$, and the equation of motion is linear.

This is the ansatz of the filament approximation.

But it assumes what we would like to prove: that $\delta\mathbf{v}$ is small.

Mercury is a poor conductor ($\sigma_{\rm Hg} \approx 10^{16}\ {\rm s}^{-1} \approx \sigma_{\rm cu}/70$), so eddy-current effects may not be too large.

Lenz' law for inductive systems suggests that changes are damped.

But, the Lorentz force is fought by incompressibility:
$\nabla\cdot\mathbf{v} = 0, \Rightarrow v_\parallel A_\perp =$ constant.

Example: radial magnetic pinch $\Rightarrow v_\perp$ should decrease.
But, longitudinal drag $\Rightarrow v_\parallel$ decreases, $\Rightarrow A_\perp$ increases,
$\Rightarrow v_\perp$ increases.

Ohm's Law

In the local rest frame of the jet, $\mathbf{J}^\star = \sigma \mathbf{E}^\star$.

$$v \ll c \;\Rightarrow \mathbf{J}^\star \approx \mathbf{J} - \rho_{\rm charge}\mathbf{v} \approx \mathbf{J}.$$

$$\Rightarrow \qquad \mathbf{E}^\star \approx \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}.$$

$$\Rightarrow \qquad \mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right).$$

$$? \quad J_x = -\frac{\sigma v_z B_y}{c} \quad ?$$

No! $J_x \Rightarrow$ surface charge buildup $\Rightarrow E_x$ that cancels $v_z B_y / c$.

[When liquid metal flows in a channel, transverse currents can be completed through the channel (Hartmann).]

For a free jet, currents flow mainly in loops about the axial velocity.

Maxwell's Equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho_{\rm charge} \approx 0, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} &\approx \frac{4\pi}{c}\mathbf{J},
\end{aligned}$$

where we ignore charge separation in the mercury,

and also ignore the displacement current (Alfven waves).

$$\Rightarrow \qquad \nabla \cdot \mathbf{J} = 0.$$

Time-independent field $\mathbf{B}_{\rm ext}$ of solenoid obeys

$\nabla \times \mathbf{B}_{\rm ext} = 0$ and $\nabla^2 \mathbf{B}_{\rm ext} = 0$ in the region of the mercury jet.

$\Rightarrow$ Induced fields $\mathbf{E}_{\rm ind}$ and $\mathbf{B}_{\rm ind}$ obey the above Maxwell equations.

Small $\sigma_{\rm Hg} \Rightarrow \mathbf{B}_{\rm ind} \ll \mathbf{B}_{\rm ext}$.

But, if set $\mathbf{B}_{\rm ind} = 0$, then $\mathbf{J} = 0$.

Magnetic Reynolds Number

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \ \Rightarrow \ \frac{\partial \mathbf{B}_{\rm ind}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_{\rm ind}) + \frac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B}_{\rm ind}.$$

Low v $\Rightarrow$ $\dfrac{\partial \mathbf{B}_{\rm ind}}{\partial t} \approx \dfrac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B}_{\rm ind}$ (diffusion equation).

For a jet of radius R, $\nabla^2 \mathbf{B}_{\rm ind} \approx \dfrac{\mathbf{B}_{\rm ind}}{R^2}$,

$$\Rightarrow \qquad \tau \approx \frac{4\pi\sigma R^2}{c^2} \approx \frac{4\pi \cdot 10^{16} \cdot (1)^2}{(3 \times 10^{10})^2} \approx 10^{-4} \text{ s}.$$

Spatial scale of $\mathbf{B}$ is the diameter D of the solenoid.

$\Rightarrow$ time scale of motion of the jet is D/v.

Magnetic Reynolds number $\mathcal{R}_M = \dfrac{v_z \tau}{D} \approx 0.01.$

$\Rightarrow$ External field fully diffused into the jet.

[Field lines are NOT "frozen in".]

$\Rightarrow$ $\mathbf{B}_{\rm ind} \ll \mathbf{B}_{\rm ext}$.

Continuous, Steady Jet on Solenoid Axis

Axial symmetry $\Rightarrow$ $v_\phi = 0.$

$$\nabla \cdot \mathbf{v} = 0 \qquad \Rightarrow \qquad \frac{1}{r}\frac{\partial r v_r}{\partial r} = -\frac{\partial v_z}{\partial z}.$$

$$\begin{aligned} v_z &\approx f_0(z) + r^2 f_2(z) + ..., \\ \Rightarrow \quad v_r &\approx -\frac{r}{2} f_0' - \frac{r^3}{4} f_2' - ... \end{aligned}$$

Don't assume $\nabla \times \mathbf{v} = 0$, as expect shear.

$$\begin{aligned} \rho(\mathbf{v}\cdot\nabla)v_r &\approx \frac{\rho r}{4}[(f_0')^2 - 2 f_0 f_0''] \\ &\quad + \frac{\rho r^3}{4}[2 f_0' f_2' - f_0 f_2'' - 2 f_2 f_0''], \\ \rho(\mathbf{v}\cdot\nabla)v_z &\approx \rho f_0 f_0' \\ &\quad + \rho r^2 f_0 f_2' \\ &\quad + \frac{\rho r^4}{2} f_2 f_2'. \end{aligned}$$

Power Series Expansion of the Pressure

$$P(r,z) \approx \rho[q_0(z) + q_2(z)r^2 + q_4(z)r^4].$$

At the free surface, $$\frac{\gamma\rho}{R} = q_0 + q_2 R^2 + q_4 R^4.$$

Better to use $\nabla \cdot \mathbf{v} = 0$,

$$\Rightarrow \Phi_z = \int_0^R v_z(r)\, 2\pi r dr \approx \pi R^2 f_0 + \frac{\pi R^4 f_2}{2} = \pi R^2(-\infty)v_z(-\infty),$$

$$\Rightarrow \qquad R^2(z) = \frac{-f_0 + \sqrt{f_0^2 + 2R^2(-\infty)v_z(-\infty)f_2}}{f_2}.$$

[Set initial value of f_2 to 10^{-10}...]

Expansion of the Solenoid Magnetic Field

$$\nabla \cdot \mathbf{B} = 0 = \nabla \times \mathbf{B}, \quad \text{plus axial symmetry} \quad \Rightarrow$$

$$B_z(r, z) = B(z) - \frac{B''(z) r^2}{4} + ...,$$

$$B_r(r, z) = -\frac{B'(z) r}{2} + \frac{B'''(z) r^3}{16} - ...,$$

where for a semi-infinite solenoid of diameter D,

$$B = B_z(0, z) = \frac{B_0}{2}\left(1 + \frac{z}{\sqrt{(D/2)^2 + z^2}}\right),$$

$$B' = \frac{dB_z(0, z)}{dz} = \frac{B_0}{2}\frac{(D/2)^2}{[(D/2)^2 + z^2]^{3/2}},$$

$$B'' = \frac{d^2 B_z(0, z)}{dz^2} = -\frac{3B_0 (D/2)^2}{2}\frac{z}{[(D/2)^2 + z^2]^{5/2}},$$

$$B''' = \frac{d^3 B_z(0, z)}{dz^3} = -\frac{3B_0 (D/2)^2}{2}\frac{(D/2)^2 - 4z^2}{[(D/2)^2 + z^2]^{7/2}}.$$

Expansion of the Current Density and Lorentz Force

$$\mathbf{J} = \frac{\sigma}{c}\mathbf{v} \times \mathbf{B}_{\rm ext}.$$

$$J_\varphi = \frac{\sigma}{c}(v_z B_r - v_r B_z) \approx \frac{\sigma}{c}\left(-\frac{r v_z B'}{2} + \frac{r^3 v_z B'''}{16} - v_r B + \frac{r^2 v_r B''}{4}\right).$$

$$F_r \left(\frac{\mathbf{J}}{c} \times \mathbf{B}\right)_r = \frac{j_\varphi B_z}{c}$$
$$\approx \frac{\sigma}{c^2}\left(-\frac{r v_z B B'}{2} + \frac{r^3 v_z (2B'B'' + BB''')}{16} - v_r B^2 + \frac{r^2 v_r B B''}{2}\right),$$

$$F_z = \left(\frac{\mathbf{J}}{c} \times \mathbf{B}\right)_z = -\frac{j_\varphi B_r}{c}$$
$$\approx \frac{\sigma}{c^2}\left(-\frac{r^2 v_z (B')^2}{4} + \frac{r^4 v_z B' B'''}{16} - \frac{r v_r B B'}{2} + \frac{r^3 v_r (2B'B'' + BB''')}{16}\right.$$

The leading term in F_r is the radial pinch on entering the magnet (or expansion on leaving), which changes the internal pressure, whose gradient leads to additional forces.

The leading term in F_z is a retarding force wherever the gradient of B is nonzero.

Expansion of the Equations of Motion

$$(f_0^2)' = -\ 2q_0', \qquad \Rightarrow \qquad q_0 = \frac{\gamma}{\rho R(-\infty)} + \frac{1}{2}v_z^2(-\infty) - \frac{1}{2}f_0^2.$$

As $q_0 = P(0, z)/\rho$, this is Bernoulli's equation for the axis of the jet, where there is no Joule heating.

$$f_0'' \approx \frac{(f_0')^2}{2f_0} + \frac{4q_2}{f_0} + \frac{\sigma}{\rho c^2}\left(BB' - \frac{f_0' B^2}{f_0}\right).$$

$$f_2'' \approx \frac{2f_0'f_2'}{f_0} - \frac{2f_2 f_0''}{f_0} + \frac{16q_4}{f_0} - \frac{\sigma}{4\rho c^2}\left(4\frac{f_2' B^2}{f_0} - 8\frac{f_2 BB'}{f_0} - 4\frac{f_0' BB''}{f_0} + BB''' + 2B'B''\right).$$

$$q_2' \approx -f_0 f_2' + \frac{\sigma}{4\rho c^2}[f_0' BB' - f_0(B')^2].$$

$$q_4' \approx -\frac{f_2 f_2'}{2} + \frac{\sigma}{32\rho c^2}[4f_2' BB' - 8f_2(B')^2 + 2f_0 B'B'' - f_0'(BB''' + 2B'B'')].$$

Status

The linearized ring and filament models predict only "modest" distortions of the jet as it passes through a 20-T magnet, although the shear is somewhat large.

Preliminary results from the nonlinear expansion seem to predict difficulty when the jet reaches distance D upstream of the magnet entrance.

Results from the FRONTIER code are eagerly awaited.

The test of Dec. 14 of a 1-mm mercury jet entering a solenoid with $B_0 = 5$ T and diameter $D = 6$ cm showed no sign of difficulty. But this test is not fully indicative of performance of a 1-cm jet in a 20-T solenoid with 30 cm diameter.

$\Rightarrow$ Need lab tests at the NHMFL, as well as continued modeling.