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Fundamental Physics During Violent Accelerations

Kirk T. McDonald
Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544

ABSTRACT

When a powerful laser beam is focussed on a free electron the acceleration of the latter is so violent that the interaction is non-linear. We review the prospects for experimental studies of non-linear electrodynamics of a single electron, with emphasis on the most accessible effect, non-linear Thomson scattering. We also speculate on the possibility of laboratory studies of a novel effect related to the Hawking radiation of a black hole.

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1. Introduction.

It is widely appreciated that the greatest laboratory acceleration obtainable with present technology is that of an electron in a high intensity laser beam. In this paper we concern ourselves with the intrinsic interest of the acceleration, rather than considering it as a step in the production of high energy particle beams. In particular we wish to draw attention to the various non-linear radiation effects which can be explored with strongly accelerated electrons.

An important qualitative feature distinguishes our understanding of the electromagnetic interaction from that of the strong, weak and gravitational interactions. Namely that the latter are fundamentally non-linear, meaning that the bosonic quanta which mediate these interactions can couple to themselves. However experimental application of our understanding is largely limited to situations in which these self-couplings can be ignored, so that the interaction is effectively linear. Yet a complete understanding of the fundamental interactions will require mastery of their non-linear aspects as well. As a step towards the elucidation of fundamental non-linear phenomena we consider the unusual case of very strong electromagnetic fields, in which non-linear effects can be induced.

We will discuss three types of non-linear electrodynamic processes. In Section 2 we speculate on the interaction of a highly accelerated electron with the zero-point energy of the electromagnetic field, in close analogy to Hawking radiation of black holes. Sections 3 and 4 review various possible effects of vacuum polarisation. In Section 5 we note that the most accessible non-linear effect is the production of higher harmonic radiation in Thomson scattering. A sketch of an experiment to detect the latter is given in Section 6.

2. A Conjecture for Future Experimental Study.

In this section we discuss a speculative effect which first aroused the author's interest in non-linear electrodynamics. This example may serve to indicate the possibilities for phenomena beyond the more standard features to be reviewed in Sections 3-5.

An appropriate point of departure is the work of Hawking (1974,1975) in which he associates a temperature with a black hole:

$$T = \frac{\hbar g}{2\pi c k}.$$

Here g is the acceleration due to gravity measured by an observer at rest with respect to the black hole, and k is Boltzmann's constant. The significance of this temperature is that the observer will find that (s)he is immersed in a bath of black-body radiation of characteristic temperature T. This is in some way due to the effect of the strong gravitational field on the ordinarily unobservable zero-point energy structure of the vacuum.

Contemporaneous with the work of Hawking several people considered quantum field theory according to accelerated observers. By the equivalence principle we might expect accelerated observers to experience much the same thermal bath as Hawking's observer at rest near a black hole. The efforts of Fulling (1972,1973), Davies (1975), and Unruh (1976) indicate that this may well be so. If a is the acceleration as measured in the instantaneous rest frame of an observer, then (s)he is surrounded by an apparent bath of radiation of temperature

$$T = \frac{\hbar a}{2\pi ck}.$$

Additional discussions of this claim are given by Sciama, Candelas and Deutsch (1981), and by Birrell and Davies (1982). 'Elementary' discussions are given by Boyer (1980), and by Donoghue and Holstein (1984).

Of experimental interest is the case when the observer is an electron. Then the electron can scatter off the bath of radiation producing photons which can be detected by inertial observers in the laboratory. This new form of radiation, which we will call *Unruh radiation*, is to be distinguished from the ordinary radiation of an accelerated electron. In particular, the intensity of radiation in the thermal bath varies as T^4. Hence we expect the intensity of the Unruh radiation to vary as $T^4 \sim a^4$. This result contrasts with the a^2 dependence of the intensity of Larmor radiation.

We illustrate this further with a semi-classical argument. The power of the Unruh radiation is given by

$$\frac{dU_{\text{Unruh}}}{dt} = \text{energy flux of thermal radiation} \times$$
$$\times \text{ scattering cross section.}$$

For the scattering cross section we take

$$\sigma_{\text{Thomson}} = \frac{8\pi}{3} r_o^2$$

where r_o is the classical electron radius. The energy density of thermal radiation is given by the usual Planck expression :

$$\frac{dU}{d\nu} = \frac{8\pi}{c^3} \frac{h\nu^3}{e^{h\nu/kT} - 1},$$

where ν is the frequency. The flux of the isotropic radiation on the electron is just c times the energy density. Note that these relations hold in the instantaneous rest frame of the electron. Then

$$\frac{dU_{\rm Unruh}}{dt d\nu} = \frac{8\pi}{c^2}\frac{h\nu^3}{e^{h\nu/kT}-1}\frac{8\pi}{3}r_o^2.$$

On integrating over ν we find

$$\frac{dU_{\rm Unruh}}{dt} = \frac{8\pi^3\hbar r_o^2}{45c^2}\left(\frac{kT}{\hbar}\right)^4 = \frac{\hbar r_o^2 a^4}{90\pi c^6},$$

using the Hawking-Davies relation $kT = \hbar a/2\pi c$.

A variation of the preceding argument has been given by Gerlach (1982). Again the key idea, taken as an assumption, is that an accelerated observer can have a non-trivial interaction with the vacuum fluctuations of the electromagnetic field. We suppose that these fluctuations, $\Delta\mathbf{E}$, lead to an additional acceleration of the observer, which for an electron yields $\langle\Delta a^2\rangle = e^2\langle\Delta\mathbf{E}^2\rangle/m^2$. The apparent strength of the vacuum fluctuations depends on the acceleration of the electron, Planck's constant, and the speed of light, but not the external electric field. By dimensional arguments, $\langle\Delta\mathbf{E}^2\rangle \sim \hbar a^4/c^7$. The Unruh radiation rate can now be estimated from the Larmor formula by inserting the fluctuation acceleration:

$$\frac{dU_{\rm Unruh}}{dt} = \frac{2}{3}\frac{e^2\langle\Delta a^2\rangle}{c^3} \sim \frac{\hbar r_o^2 a^4}{c^6}.$$

A numerical comparison with Larmor radiation is instructive:

$$\frac{dU_{\rm Unruh}}{dt} \sim 4.1\times 10^{-118} a^4 \qquad \text{(in c.g.s. units)}$$
$$\frac{dU_{\rm Larmor}}{dt} \sim 5.7\times 10^{-51} a^2.$$

The two radiation effects are comparable for $a \sim 3\times 10^{33}{\rm cm/sec}^2 \sim 3\times 10^{30} g$, where g is the acceleration due to gravity at the surface of the earth. If this acceleration is to be provided by an electric field we then need

$$E \sim 2\times 10^{17} \text{ volts/cm}.$$

This is about one order of magnitude larger than the 'critical' field $m^2c^3/e\hbar$ to be discussed in Section 4b. The consequence is that the Unruh radiation effect will manifest itself only in the context of other non-linear electrodynamic phenomena. Untangling the various features of radiation when $E > 10^{17}$ volts/cm will be a formidable challenge. For example, the radiation will include e^+e^- pairs as well as photons.

Note that $E_{\rm lab}$ need not be $\sim 3\times 10^{17}$ volts/cm if a relativistic electron probes the field. For a 50 GeV electron $\gamma \sim 10^5$ so that $E_{\rm lab} \sim 3\times 10^{12}$ volts/cm will suffice. This is still large by present standards but far short of the fundamental limitation for laboratory field strengths to be discussed in Section 4d.

The Unruh radiation effect is purely electrodynamic in origin, and hence should be contained in a complete theory of quantum electrodynamics. This is claimed to be proven true in the work of Myhrvold (1983), but the form of his argument does not shed immediate light on experimental considerations.

The possibility of observing the effect of Unruh radiation in another context has been considered by Bell and Leinaas (1983). Salam and Strathdee (1977), Barshay and Troost (1977), and Hosoya (1978) have considered the possible relevance of the Hawking-Davies temperature to thermodynamic models of the strong interaction.

3. Light by Light Scattering.

A fundamental non-linearity in electrodynamics is light by light scattering. It has not yet been observed directly. This process is expected to occur in higher order of perturbation theory due to the production of a virtual e^+e^- pair. A related non-linear process is two photon production of an electron-positron pair,† $\omega\omega \to e^+e^-$. The cross sections for these processes are not particularly small‡:

$$\sigma_{\omega\omega\to\omega\omega} \sim \frac{\alpha^4}{\omega_1\omega_2} \qquad \text{if } \omega_1\omega_2 \gtrsim m^2 \qquad \text{(Achieser, 1937)}$$
$$\sim \frac{10^{-30}}{\omega_1\omega_2}\ \text{cm}^2 \qquad \text{for } \omega_1, \omega_2 \text{ in MeV.}$$

Thus we might expect one light by light scattering if two beams of 10^{15} photons of energy 1 MeV are brought into collision. As laboratory beams in the MeV range have typically $\lesssim 10^{10}$ photons it has not been practical to search directly for light by light scattering.

Mikaelian (1982) has suggested colliding 50 Gev photons against a laser beam of 5 eV photons. Then $\omega_1\omega_2 \sim m^2$ so that $\sigma \sim 10^{-30}$ cm^2. A one joule laser pulse would contain $\sim 10^{18}$ photons of 5 eV, so we might expect one scattering for every 10^{12} high energy photons. While the prospects for this experiment are not immediate it is conceivable it will become possible within the next decade.

Indirect observation of light by light scattering effects has been made at the PEP and PETRA electron-positron storage rings via the reaction $e^+e^- \to e^+e^- + X$. Each electron radiates a photon which then collide yielding final state X.

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† We use ω to symbolize a photon, as well as its energy. The symbol γ will be reserved for Lorentz transformations.

‡ In making estimates of cross sections and reaction rates it is convenient to use units in which $\hbar = c = 1$. Then in dimensional arguments we have
cross section $= \sigma \sim$ length$^2 \sim 1/$energy^2
rate $\sim 1/$time $\sim 1/$length $\sim$ energy
rate/volume $\sim$ energy^4.
To obtain numerical estimates we insert appropriate powers of $\hbar$ and c, noting
$\hbar c \sim 2 \times 10^{-11}$MeV cm; $\qquad \hbar \sim 6.6 \times 10^{-22}$MeV sec.
Of course, $\alpha = e^2$ $(= e^2/\hbar c) \sim 1/137$, and the electron rest energy $= m$ $(=\ mc^2) \sim 0.5$ Mev.

The cross section for pair production by two photons, $\omega\omega \to e^+e^-$, is larger than than for $\omega\omega \to \omega\omega$ by $(1/\alpha)^2$ (Breit and Wheeler, 1934):

$$\sigma_{\omega\omega\to e^+e^-} \sim \frac{\alpha^2}{\omega_1\omega_2} \qquad \text{if } \omega_1\omega_2 \gtrsim m^2.$$

This process could be observed with present techniques, as discussed in Section 4c.

4. Multiphoton Effects.

The development of the laser has fostered fruitful growth of the field of non-linear optics. The basic effect was first observed by Franken et al. (1961). The motion of atomic electrons in an intense laser beam must be described by an anharmonic potential. The forced oscillations in this potential include higher harmonic components. Equivalently, two or more photons are absorbed by the electron before the absorbed energy is radiated away as a single higher frequency photon. In the last 20 years a rich variety of related phenomena has emerged, little anticipated at the time of the initial explorations into non-linear optics.

In the 1960's considerations of non-linear effects were extended to the case of a free electron illuminated by a laser beam. The rest of this paper arises out of these considerations.

a. Relativistic effects.

Multiphoton effects become important in the electrodynamics of a single electron when the fields are strong. There are two different measures of how strong the field must be, applicable to two different classes of phenomena. The first criterion is classical and applies to wave fields. The field is strong when the potential difference across one wavelength is greater than the rest energy of the electron. We define

$$\eta = \frac{eE/k}{mc^2} = \frac{eE}{m\omega c} \ \left(= \frac{eE}{m\omega} \right)$$

where E is the electric field strength, ω is the wave frequency, and k is the wave number. This is a relativistic invariant†:

$$\eta = \frac{e}{mc^2}\sqrt{-A_\mu A^\mu} \qquad \text{where } A_\mu = 4 - \text{vector potential.}$$

When $\eta \gtrsim 1$ an electron in the wave field takes on velocities comparable to c during its oscillatory motion. In this case higher multipole radiation is emitted with comparable intensity to dipole radiation. In other words, several photons are absorbed before a higher frequency photon is emitted. The higher harmonic radiation rate is non-linear in the field strength. This phenomenon should be readily observable with present technology. We defer detailed discussion until Sections 5 and 6. The remainder of this section concerns non-linear effects associated with the production of e^+e^- pairs.

† We use a metric such that $A_\mu A^\mu = A_0^2 - \mathbf{A}^2$.

b. Spontaneous pair creation.

Another criterion is that the field is strong when the potential difference across one Compton wavelength of an electron is greater than the rest energy of the electron. We define

$$\xi = \frac{eE \cdot \hbar/mc}{mc^2} = \frac{\hbar eE}{m^2c^3} \left(= \frac{eE}{m^2}\right) = \frac{E}{E_{\rm critical}}.$$

It is interesting to note the value of the field strength when $\xi = 1$:

$$E_{\rm critical} = \frac{m^2c^3}{e\hbar} \left(= \frac{m^2}{e}\right) \sim 1.3 \times 10^{16} \text{ volts/cm}.$$

Note that ξ is not a relativistic invariant so that it must be invoked with care. However for a static, purely electric field ξ is well defined. Then if $\xi \gtrsim 1$ the static field is unstable against spontaneous production of e^+e^- pairs, certainly a non-linear effect (Schwinger, 1951, 1954; Brezin and Itzykson, 1970). We estimate the rate of spontaneous pair production per cm^3 as:

$$\text{rate/cm}^3 \sim \alpha^2 \times \text{ field energy density} \times \text{'barrier penetration factor'}.$$

For the 'barrier penetration factor' we adopt the picture that virtual e^+e^- pairs are bound in the vacuum by energy $2m$. In a field E the e^+ and e^- must be moved distance $\pm m/eE$ to overcome the 'binding'. As a measure of the probability of this occurence we take the ratio of the squares of the (non-relativistic) wave function at distances 0 and $2m/eE$ in the potential eEx:

$$\text{factor} = \exp\left(-2 \int_0^{\frac{2m}{eE}} \sqrt{2m(2m - eEx)}dx\right) = e^{-\frac{16m^2}{3eE}} = e^{-\frac{16}{3\xi}}.$$

(See, for example, Landau and Lifshitz, Quantum Mechanics, secs. 46 and 77.) It turns out the the coefficient 16/3 should be more like 8/3. Then

$$\begin{aligned} \text{rate/cm}^3 &\sim \alpha^2 E^2 \; e^{-\frac{8}{3\xi}} \sim \alpha m^4 \xi^2 \; e^{-\frac{8}{3\xi}} \\ &\sim 10^{50} \xi^2 \; e^{-\frac{8}{3\xi}} \text{ pairs/sec/cm}^3. \end{aligned}$$

To have a rate of $1/\text{sec/cm}^3$ requires $\xi \sim 1/40$ or $E \sim 3 \times 10^{14}$ volts/cm. The strongest static electric fields produced in the laboratory are perhaps 10^5 volts/cm. Thus spontaneous pair production rates will be low.

c. Stimulated pair creation.

Spontaneous pair production is not possible in a plane wave field as energy-momentum conservation would not be satisfied. However if the wave field is probed by another wave (or particle) stimulated emission of an e^+e^- pair can occur. For example, consider the reaction

$$n\omega + \omega' \to e^+e^-.$$

Here n laser photons collide with a high energy photon ω' to produce the pair. This may be thought of as a multiphoton version of the Breit-Wheeler process, and was anticipated by Toll (1952). To estimate the rate we first consider the process in the rest frame of the e^+e^- pair. In this frame the probe photon has energy m, so the Lorentz boost from the lab frame is $\gamma \sim \omega'/2m$, assuming $\omega' \gg m$. Hence the electric field strength of the laser beam in the e^+e^- rest frame is $\omega' E/m$, supposing the photons ω and ω' move in opposite directions. Comparing with the case of spontaneous emission we infer that parameter

$$\chi = \frac{\omega' E/m}{E_{\text{critical}}} = \frac{\omega\omega'}{m^2}\frac{eE}{m\omega} = \frac{\omega^\mu \omega'_\mu}{2m^2}\eta$$

will govern the pair production rate. Note that χ is a Lorentz invariant. Furthermore in the e^+e^- rest frame we estimate

$$\text{rate} \sim \alpha m^4 \chi^2 \;\; e^{-\frac{8}{3\chi}} \times \text{effective volume probed by photon } \omega'.$$

We estimate the probed volume as only $\lambda^3_{\text{probe}} \sim 1/m^3$, evaluating λ_{probe} in the e^+e^- rest frame. Then

$$\text{rate in rest frame} \sim \alpha m \chi^2 \;\; e^{-\frac{8}{3\chi}}$$

and hence

$$\begin{aligned} \text{rate in lab} &\sim \frac{1}{\gamma} \times \text{rate in rest frame} \\ &\sim \alpha \frac{m^2}{\omega'} \chi^2 \;\; e^{-\frac{8}{3\chi}} \quad \text{pairs/sec/probe photon.} \end{aligned}$$

The more exact result is

$$\begin{aligned} \text{rate} &\sim \frac{3}{8}\alpha \frac{m^2}{\omega'} \left(\frac{\chi}{2\pi}\right)^{3/2} \;\; e^{-\frac{8}{3\chi}} \\ &\sim 10^{19} \frac{m}{\omega'} \left(\frac{\chi}{2\pi}\right)^{3/2} \;\; e^{-\frac{8}{3\chi}} \quad \text{pairs/sec/probe photon.} \end{aligned}$$

If we use a probe photon of 50 GeV then $\omega'/m \sim 10^5$. The Omega Nd:glass laser of the U. of Rochester (Seka *et al.* 1980) is quoted as achieving fields equivalent to $\eta \sim 3$. The 1 μm photon energy is $\omega/m \sim 10^{-5}$. If such a laser could be brought to a 50 GeV photon beam then we would have $\chi \sim 3$, and the pair production rate would be $\sim 10^{23}$ per sec per probe photon. That is, each probe photon would have essentially unit probability of producing an e^+e^- pair.

Thus stimulated pair creation is observable with present technology. One must merely overcome the logistical difficulty of bringing a high intensity laser beam near a high energy photon beam

For $\chi \gg 1$ our simple rate estimate should be modified according to the detailed QED calculations of Reiss (1962); Nikishov and Ritus (1964, 1965, 1967); and Narozhny, Nikishov and Ritus (1965).

It is instructive to continue our considerations of the process $n\omega + \omega' \to e^+e^-$. We can calculate the number n of photons absorbed from the laser beam. In the e^+e^- rest frame the total energy of these photons must also be m[†]: $n\omega^\star = m$. Now

$$\omega^\star \sim 2\gamma\omega = \frac{\omega\omega'}{m} \qquad \text{and hence} \qquad n = \frac{m^2}{\omega\omega'}.$$

Thus we have

$$\chi = \frac{\omega\omega'}{m^2}\eta = \frac{\eta}{n}.$$

For a significant pair production rate the laser field strength must satisfy $\eta \gtrsim n = \omega\omega'/m^2$. The weakest laser field that gives signficant pair production is such that $\eta \gtrsim 1$ (provided also that $\omega\omega' \gtrsim m^2$). In this case only one photon is absorbed from the laser beam, exactly as in the process studied by Breit and Wheeler. It is notable that the classical criterion for strong fields, $\eta \gtrsim 1$, must be satisfied in practice before the quantum non-linear effects will be observable.

We can verify that our present rate estimate is consistent with the Breit-Wheeler cross section discussed in section 1. When $\chi \sim 1$ we have

$$\text{rate} \sim \frac{\alpha m^2}{\omega'} \sim \alpha\omega\eta \ \text{ pairs/sec/probe photon.}$$

But

$$\text{rate} = \sigma \times \text{flux of laser photons} = \sigma\frac{E^2 c}{8\pi\hbar\omega} \sim \sigma\frac{E^2}{\omega}.$$

Thus

$$\sigma \sim \frac{\alpha\omega^2\eta}{E^2} = \frac{\alpha^2}{m^2\eta}.$$

If in addition $\eta \sim 1$, which implies $n = 1$ also, then

$$\sigma \sim \frac{\alpha^2}{\omega\omega'} \sim \sigma_{\text{Breit-Wheeler}}.$$

Another measure of the reaction rate is the mean free path of the probe photon before pair

† We use the superscript $\star$ to indicate quantities evaluated in the rest frame.

production occurs in the laser beam. Again we consider the case when $\chi \sim 1$, and again

$$\text{rate} \sim \frac{\alpha m^2}{\omega'} \sim \alpha\omega\eta \text{ pairs/sec.}$$

Then

$$\text{mean free path} \sim \frac{c}{\text{rate}} \sim \frac{c}{\alpha\omega\eta} \sim \frac{\lambda}{\alpha\eta}.$$

If $\eta \sim 1$ also, then

$$\text{mean free path} \sim 137\lambda.$$

Stating this yet another way, when the pair production rate becomes significant ($\chi \sim 1,\ \eta \sim 1$) then the probability of pair production is roughly α per wavelength of laser beam traversed by the probe.

d. A fundamental limit to laser field strength.

Consider a laser beam focussed by a lens. The field is no longer a plane wave and now 'spontaneous' pair production is possible in the absence of a probe particle. Photons which are converging on the focus can be transformed to a frame in which they collide head on, and so pair production is kinematically allowed.

If θ = half angle of convergence at the focus, then we find that the critcal field strength for pair production is

$$E_{\text{critical}} \sim \frac{m^2}{e\sin\theta}.$$

If the focus is diffraction limited the converging photons cannot be effectively in collision outside a volume much larger than λ^3. Nor will the strong field region be much larger than this. To estimate the fraction of the beam energy lost in spontaneous pair creation we return to the rate expression for static fields. If $E \sim E_{\text{critical}}$ at the focus,

$$\text{rate} \sim \alpha^2 E^2 \text{ pairs/sec/cm}^3.$$

Then

$$\Delta U_{\text{lost}} \sim m \times \text{rate} \times \frac{\lambda}{c} \times \lambda^3 \sim \frac{\alpha^2 m}{\omega} E_{\text{crit}}^2 \lambda^3 \sim \frac{\alpha^2 m}{\omega} U_{\text{beam}},$$

and

$$\frac{\Delta U}{U} \sim \frac{\alpha^2 m}{\omega}.$$

For laser photons of energy less than 25 eV, $\Delta U/U \to 100\%$. In this sense

$$E_{\text{critical}} \sim \frac{m^2}{e\sin\theta} \sim \frac{1.3 \times 10^{16}}{\sin\theta} \text{ volts/cm}$$

represents a fundamental limit to the field strength obtainable in focussed laser beams.

e. Restrictions on stimulated pair creation.

In principle the preceding analysis could be applied to production of pairs of any charged particle, not just the electron. Consider the attempt to produce a pair of particles of mass M in a strong field by probing with a photon ω'. For a significant rate we need, according to the preceding,

$$\chi = \frac{\omega' E/M}{M^2/e} \sim \frac{\omega' m^2}{M^3} \sim 1,$$

noting that $E \sim m^2/e$ will be the strongest achievable electric field. Thus

$$M^3 \sim \omega' m^2.$$

Even with $\omega' = 500$ GeV $\sim 10^6 m$ we can produce only $M \sim 100m$. However the next lightest particle after the electron is the muon with $M_\mu \sim 206m$.

A similar limitation obtains if we collide two laser beams against each other. The number of photons which can be extracted from a beam to produce a particle of mass M is roughly $\eta = eE/M\omega$. The energy of these photons is then eE/M, so we must have field strength $E \sim M^2/e$. But as we are limited to $E \lesssim m^2/e \sin\theta$, the only possibility is $M \sim m_{\text{electron}}$.

Thus it seems likely that future studies of pair production by intense laser beams will be limited to electron-positron pairs only.

5. Multiphoton Effects in Thomson Scattering.

As suggested in Section 4a the most accessible non-linear effect in the interaction of electrons and photons involves higher harmonic radiation in Thomson scattering:

$$n\omega + e \to \omega' + e'.$$

a. Classical analysis.

For electric field strengths of the incident wave less than the critical field m^2/e (in the rest frame of the electron) the interaction is essentially classical. It is useful to think of the scattering process in terms of multipole radiation. The motion of the electron in the wave field generates all orders of multipole moments, but the radiation due to the nth order moment is smaller than dipole radiation by a factor of order $(v/c)^{2n-2}$. [See Dyson (1979) for an interesting application of this memorable fact.] As the electron's velocity in the wave field approaches c the intensity of multipole radiation at higher harmonic frequencies becomes comparable to that of dipole radiation.

The case of a circularly polarised incident wave is simplest to analyze. In the frame in which the electron has no motion in the direction of the wave, the electron moves in a circle with angular velocity ω equal that of the wave frequency. Thus $eE = m\omega^2 r = m\omega v$ and

$$\frac{v}{c} = \frac{eE}{m\omega c} = \eta.$$

Strictly speaking, for $\eta \gtrsim 1$ we should write

$$eE = \gamma m\omega v \qquad \text{where} \qquad \gamma = 1/\sqrt{1-\beta^2} \qquad \text{and} \qquad \beta = v/c.$$

Then

$$\gamma\beta = \eta \qquad \text{so that} \qquad \gamma = \sqrt{1+\eta^2} \qquad \text{and} \qquad \beta = \eta/\sqrt{1+\eta^2}.$$

Also, the radius of the circle is

$$r = \frac{eE}{\gamma m\omega^2} = \frac{\eta}{\sqrt{1+\eta^2}}\frac{\lambda}{2\pi} \leq \frac{\lambda}{2\pi}.$$

For waves with other than circular polarisation it is convenient to define

$$\eta^2 = \frac{e^2\langle E^2\rangle}{m^2\omega^2c^2},$$

where the average is with respect to time. Then expressions involving η deduced for the case of circular polarisation will hold for all types of polarisation.

In any case we see that the condition $v/c \sim 1$ for a significant rate of higher harmonic radiation becomes $\eta \gtrsim 1$. Then the cross section for scattering into frequency $n\omega$ varies approximately as

$$\sigma_{n\omega} \sim \sigma_{\text{Thomson}}\eta^{2n-2} \sim r_o^2\eta^{2n-2} \qquad \text{for } \eta \lesssim 1.$$

These arguments can be expressed in terms of photons by noting that radiation at frequency $n\omega$ corresponds to absorption of n photons by the electron followed by emission of one photon at frequency $n\omega$. A simple QED estimate of the cross section would be

$$\sigma_{n\omega} \sim \frac{\alpha^{n+1}}{m^2} = \alpha^{n-1}r_o^2$$

counting one power of α for each external photon, and noting $r_o = \alpha/m$. On comparison with our classical argument we reach an important conclusion. In strong fields where a QED approach is necessary ($E > m^2/e$), the appropriate dimensionless QED expansion parameter which depends on e^2 is not α but rather η^2. Thus we anticipate that a detailed QED analysis of higher harmonic radiation in the regime $\eta \gg 1, E > m^2/e$ cannot be done via Feynman diagrams, but must involve 'non-perturbative' techniques.

b. The mass shift effect.

If we prefer to think of the non-linear Thomson scattering in terms of photons we must take account of an effect which may seem surprising, although it is readily understood classically. When illuminated by the wave field, the electron undergoes oscillatory motion. The frequency of oscillation is that of the incident photons, and the amplitude of the oscillation is less than a wavelength of the photons. Thus the incident wave cannot resolve the details of the oscillatory motion (although a higher frequency probe could indeed trace the classical path of the electron). The scattering process can be affected by this motion only in an average way. The physical effect can be summarized by saying that the effective mass of an electron in a circularly polarised wave is

$$\overline{m} = \gamma m = m\sqrt{1+\eta^2}.$$

Recall that we are in a frame in which the average velocity of the electron is zero.

By effective mass we mean that the rapid oscillatory motion which causes the mass increase cannot be resolved by the scattering process, so that in effect we are dealing with a heavy electron which doesn't oscillate. In kinematic calculations we must use an effective 4-vector for the electron with invariant mass $\overline{m}$ rather than m, as described in the next section.

c. The drift velocity.

An important subtlety should be noted (Brown and Kibble, 1964; this is overlooked in such treatments as Landau and Lifshitz, Classical Theory of Fields, secs. 47-48). If the electromagnetic wave is incident on a free electron the subsequent motion of the electron is not purely oscillatory. The electron also takes on a 'drift' velocity along the direction of propagation of the wave, which can be substantial for $\eta \gtrsim 1$. The drift velocity is the result of transient effects when the electron first encounters the field. During this time the electron motion is not perfectly in phase with the wave and the $\mathbf{v} \times \mathbf{B}$ force has a component in the direction of the wave propagation.

We can avoid detailed consideration of the transient behavior and calculate the drift velocity of the electron once steady motion is established by thinking of the reaction in terms of photons. The argument which led to the relation $\overline{m} = m\sqrt{1+\eta^2}$ was made in the frame in which the drift velocity of the electron vanishes. However this frame is not necessarily the lab frame. In the general case suppose p_μ is the 4-momentum vector of the electron in the lab frame before entering the wave, and ω_μ is the 4-vector of a photon in the lab. Then $\overline{p}_\mu$, the effective 4-vector of the electron in the wave, must have the form

$$\overline{p}_\mu = p_\mu + \epsilon\omega_\mu.$$

We know that

$$\overline{p}^2 = \overline{m}^2 \equiv m^2 + \eta^2 m^2$$

Hence

$$\epsilon = \frac{\eta^2 m^2}{2p_\mu\omega^\mu}.$$

For example, if the electron is initially at rest,

$$p_\mu = (m, 0, 0, 0), \qquad \omega_\mu = (\omega, 0, 0, \omega),$$

so that

$$p_\mu \omega^\mu = m\omega \qquad \text{and} \qquad \epsilon = \frac{\eta^2 m}{2\omega}.$$

Then

$$\overline{p}_\mu = \left(m\left(1 + \frac{\eta^2}{2}\right), 0, 0, \frac{\eta^2 m}{2} \right).$$

The drift velocity is thus

$$\beta_\| = \frac{\eta^2}{2 + \eta^2}.$$

d. The frequency of the scattered light.

For $\eta \gtrsim 1$ the laboratory motion of an electron initially at rest becomes relativistic, which causes further changes in the appearance of the scattering process. To quantify this change we calculate the kinematics of the multiphoton reaction

$$n\omega + e \to \omega' + e'.$$

In terms of 4-vectors this is

$$n\omega_\mu + \overline{p}_\mu = \omega'_\mu + \overline{p}'_\mu.$$

Noting that $\overline{p}'^2 = \overline{m}^2$ we find

$$n\omega'_\mu \omega^\mu + \omega'_\mu \overline{p}^\mu = n\omega_\mu \overline{p}^\mu.$$

For example, if the electron is initially at rest and we write

$$\omega'_\mu = (\omega', \ \omega' \sin\theta, \ 0, \ \omega' \cos\theta)$$

then we find

$$\omega' = \frac{n\omega}{1 + \left(\frac{n\omega}{m} + \frac{\eta^2}{2}\right)(1 - \cos\theta)}.$$

This is the appropriate form of the Compton scattering relation for high field strengths.

If we scatter optical photons off electrons then $\omega/m \lesssim 10^{-5}$ which is negligible as usual. But η^2 is large when there is significant probability for higher harmonic radiation. In this case there is substantial variation of the frequency ω' with scattering angle. This effect occurs for the fundamental harmonic ($n = 1$) as well. Measurement of the frequency shift with angle allows a direct determination of the parameter η^2, independent of the intensities of the various higher harmonics. This feature will aid greatly in the interpretation of experimental results.

In the mid 1960's a small controversy arose as to the observability of the electron mass shift and of the frequency shift of the scattered photon. Although these are essentially classical effects, there is some difficulty in demonstrating them in a QED approach. This arises because QED calculations for strong fields can only be made at present for plane waves of infinite extent and duration. Reasonably convincing arguments show that the QED calculations indeed have the proper classical limit in the case of pulsed fields (Kibble, 1966b; Eberly and Sleeper, 1968). See also Eberly (1969) for a review with extensive references. The conclusion remains that the photon frequency shift should be detectable, even if its value is not exactly that given by arguments based on plane waves.

e. The possibility for experiment.

The program of a possible experiment is now reasonably clear. The key is the production of a laser beam for which the field strength satisfies $\eta = eE/m\omega c \gtrsim 1$. We anticipate that in practice this can be obtained only by focussing the beam, and that for a diffraction limited focus the condition $\eta \sim \eta_{\rm max}$ can be maintained only over a volume $\sim \lambda^3$. The beam then scatters off any electrons in this volume leading to a discrete spectrum of radiation at any fixed angle. The fundamental scattered frequency will be lower than the laser frequency. The experiment should detect this frequency shift, as well as measure the intensity of the various harmonics of the scattered light.

A rate estimate for the non-linear Thomson scattering experiment can be made by recalling the Larmor formula,

$$\frac{dU}{dt} \sim \frac{e^4 E^2}{m^2}.$$

The energy radiated in one cycle of the wave is

$$dU \sim \frac{e^4 E^2}{m^2 \omega} \text{ per cycle},$$

and the number of photons radiated is

$$dN = \frac{dU}{\omega} \sim \frac{e^4 E^2}{m^2 \omega^2} \sim \alpha \eta^2 \text{ photons/cycle}.$$

This rate is of course to be multiplied by the number of electrons in volume λ^3. Thus once fields with $\eta \sim 1$ have been achieved the scattering rate is quite substantial. [Strictly speaking the preceeding argument holds in the average rest frame of the electron. The conclusion is clearly invariant of the choice of frame.]

Non-linear effects have been observed in the scattering of light from a CO_2 laser or a Nd:glass laser off an electron plasma. See Burnett *et al.* (1977) and McLean *et al.* (1977). Up to the 46th harmonic has been observed by Carman *et al.* (1981). In these experiments the laser beam induces a sharp density gradient on the plasma which then supports non-linear plasma oscillations leading to higher harmonic radiation. This effect is to be distinguished from present concerns of scattering off a single electron. In particular the electron mass shift and attendant frequency shift of the scattered photon have not been observed in the plasma experiments.

f. The need for relativistic electrons.

In practice it is doubtful that a free electron which is initially at rest can occupy the focus of an intense laser beam. The electron will be expelled from the strong field region by the 'ponderomotive' or 'field-gradient' force (Kibble, 1966a,b). A sense of this force can be gotten from a non-relativistic argument. The effective mass $\overline{m}$ can be thought of as describing an effective potential for the electron inside the wave field.

$$U_{\text{eff}} = \overline{m}c^2 = mc^2\sqrt{1+\eta^2} \sim mc^2 + \frac{1}{2}mc^2\eta^2.$$

Hence

$$\mathbf{F} = -\nabla U \sim -\frac{1}{2}mc^2\nabla\eta^2 = -\frac{4\pi e^2}{m\omega^2 c}\nabla I$$

where I is the intensity of the wave. In a focussed laser beam the intensity has a strong transverse gradient which will push an electron away from the optical axis.

Another view of this effect is obtained by analogy to the reflection of low frequency light off of an electron plasma. From the dispersion relation for light in a plasma,

$$\omega^2 = k^2c^2 + \omega_p^2 \qquad \text{where } \omega_p = \text{plasma frequency},$$

we infer that a photon inside the plasma has an effective mass given by

$$m_{\text{eff}}^2 = \left(\hbar\omega_p/c^2\right)^2.$$

For an electron in an intense photon beam we found a mass shift of $\Delta m^2 = \eta^2 m^2$. If we consider the laser beam as a kind of 'plasma' of photons then we see that the quantity which plays the role of the plasma frequency is $\eta mc^2/\hbar$. Just as photons with $\omega < \omega_p$ can't penetrate an electron plasma, we infer that electrons with momentum less that ηmc can't penetrate a photon beam. Thus electrons with initial velocities such that $\gamma\beta < \eta$ will be expelled from the strong field region of the laser beam.

The conclusion is that intense laser beams can only be probed by relativistic electrons.

g. Very strong fields.

We remark briefly on the case when the wave field is so strong that $\eta \gg 1$. Consider a circularly polarised wave of frequency ω incident head on with an electron with Lorentz factor γ. If the electron is not to be badly deflected by the wave then we need $\gamma \gg \eta$ according to the preceding argument. Following an analysis like that of Section 5c we find that the energy of the electron once inside the wave remains γm to first order in η/γ. Of course much of this energy is in the transverse motion of the electron, so its longitudinal velocity has been decreased. This is described by the effective mass according to $\gamma m = \gamma_{\|}\overline{m}$. Since $\overline{m} = m\sqrt{1+\eta^2} \sim m\eta$ we have that $\gamma_{\|} \sim \gamma/\eta$. On transforming to the average rest frame of the electron the wave has frequency $\omega^\star \sim 2\gamma_{\|}\omega \sim 2\gamma\omega/\eta$. In this frame the electron is not at rest but moves in a circle at relativistic velocities, described by the Lorentz factor

$\gamma_\perp = \overline{m}/m \sim \eta$. The radiation of the circling electron can be thought of as synchrotron radiation. From a classical analysis of the latter we know that the spectrum peaks at frequency $\omega^{\star\prime} \sim \gamma_\perp^3 \omega^\star \sim \eta^3 \omega^\star$, corresponding to maximum strength for the harmonic with $n \sim \eta^3$. This radiation is strong only in the plane of the circular motion (in the average rest frame of the electron). On transforming back to the lab frame, the radiation has characteristic frequency $\omega' = \gamma_\| \omega^{\star\prime} \sim \gamma\eta^2\omega$, and lies in a narrow cone of half angle $\sim \eta/\gamma$.

For strong enough wave fields the resulting radiation will include e^+e^- pairs and a classical analysis no longer suffices. The threshold for this QED correction is that the energy of the radiated photon as viewed in the average rest frame of the electron is equal to $\overline{m}$. The critical energy is $\overline{m}$ and not m as any pairs produced must have the energy of the transverse oscillations of electrons in the strong field. The condition $\omega^{\star\prime} \gtrsim \overline{m}$ transforms to lab frame quantities as $\gamma\eta\omega \gtrsim m$ or $\gamma E \gtrsim m^2/e$ recalling that $\eta = eE/m\omega$. Not surprisingly, this is just the condition that the electric field in the true rest frame of the electron be stronger than the critical field introduced in Section 4b. Then the electron probes the wave field so as to stimulate pair creation as discussed in Section 4c.

There is of course another limit in which QED corrections are important. Namely when $\gamma\omega \gtrsim m$ the Thomson scattering of a single photon off an electron goes over to Compton scattering. This possibility is unrelated to the intensity of the photon beam and is not a non-linear effect.

Detailed QED calculations of the higher harmonic spectrum are available for the case of a plane wave of arbitrary intensity incident on an electron. However, only when $\gamma E \gtrsim m^2/e$ or $\gamma\omega \gtrsim m$ will the results depart significantly from those of a classical calculation. The results are given in a rather complicated form and will require numerical integration in order to be compared with a specific experiment. The case of circular polarisation is somewhat simpler, and has been considered by Goldman (1964); Narozhny, Nikishov and Ritus (1965); Nikishov and Ritus (1965,1967) (see also Landau and Lifshitz, Relativistic Quantum Theory, sec. 98). The case of linear polarisation is treated by Brown and Kibble (1964), and Nikishov and Ritus (1964). Earlier considerations include Sengupta (1949,1952), Fried (1961,1963), and Vachaspati (1962).

The calculations are based on the exact solution of the Dirac equation for an electron in a plane wave (Volkov, 1935; Schwinger, 1951,1954). The radiated photon is treated as a perturbation, leading to only one power of α in the results. Some of the authors give series expansions suitable for the limits $\eta \ll 1$ or $\eta \gg 1$, which are not of present interest. In comparison of these results with experiment one must also consider how well a focussed laser beam approximates a plane wave.

g. Comparison with radiation in an undulator.

The phenomenon of non-linear Thomson scattering is closely related to the production of higher harmonic radiation in the passage of an electron through a static but spatially oscillating magnetic field (Motz, 1951). In the analysis of an 'undulator' or 'wiggler' magnet of periodicity λ_o one introduces the dimensionless parameter $\eta = eB\lambda_o/2\pi mc^2$, where B is the r.m.s. spatial average magnetic field. For example, the radiation emitted in the forward direction has wavelength $\lambda \sim 2\lambda_o/\gamma_\|^2 = 2\lambda_o(1+\eta^2)/\gamma^2$ where γ is the Lorentz factor of the incident electron. Although the total velocity of the electron is unaffected by the undulator, its longitudinal component is reduced to compensate for the transverse oscillations. This leads to the appearance of η in the expression for λ. For $\eta \gtrsim 1$ the oscillations are strong enough that higher harmonic radiation is probable. The higher harmonic radiation has been observed (Billardon *et al.*, 1983), and is considered a background to the operation of undulators as free electron lasers, where it is desired to amplify only one frequency. Coherent production of higher harmonic radiation has been observed by Girard *et al.* (1984) from a bunched electron beam obtained in an 'optical klystron.'

The magnetic field of the undulator can be thought of as consisting of 'virtual photons.' In the lab frame these carry momentum h/λ_o but no energy. However in the average rest frame of a relativistic electron inside the undulator the virtual photons appear very much like real photons. If we wish to describe the radiation process as a kind of Thomson scattering off the virtual photons we must note that the electron takes on an effective mass $\overline{m} = m\sqrt{1+\eta^2}$ inside the undulator, so that $\gamma m = \gamma_{\|}\overline{m}$. As for the case of an electron in a laser beam, the 'mass shift' effect is an artifice of a description which emphasizes the longitudinal motion of the electron and averages over the transverse oscillations.

6. Sketch of an Experiment.

We sketch an experiment which might be the beginning of a program to demonstrate the various non-linear electrodynamic effects discussed in this paper. The general technique is to bring a very intense laser beam into collision with a highly relativistic electron beam. The eventual goal is that $\gamma E > 10^{17}$ volts/cm, where γ is the Lorentz factor for the electron beam and E is the (focussed) laser electric field strength. A study of non-linear Thomson scattering would provide a good first opportunity to combine the needed laser and particle accelerator technologies.

a. An experiment with MeV electrons.

To produce the non-linear Thomson effect the laser field strength should satisfy $\eta = eE/m\omega c \gtrsim 1$. This field strength is presently available in CO_2 lasers such as the Antares facility at Los Alamos, and the 1 p-sec laser of Corkum (1983), and in Nd:glass lasers such as the Omega facility at the U. of Rochester (Seka *et al.*, 1980). The electron beam should have $\gamma\beta \gg \eta$ in order not to be badly deflected by the laser beam as noted in Section 5g. To match lasers with field strength $\eta \sim 1$ the electron beam energy need only be a few MeV. Thus it seems suitable to bring a small electron accelerator to an existing laser for the initial experiment.

When the laser beam scatters off the relativistic electrons the light will be doppler shifted to higher frequencies. For an electron beam with $\gamma \sim 5-10$ it can be arranged that the light scattered from a CO_2 laser will attain optical frequencies, which is excellent for detection of weak signals. In case of the higher frequency Nd:glass lasers the result of the doppler shift is to place most of the scattered light in the far ultraviolet, which is less suitable. Hence we will presume the use of a CO_2 laser for the non-linear Thomson scattering experiment.

We give some details of the doppler shift effect. Suppose the laser beam and the electron beam meet and angle α, where $\alpha = 0°$ for a head-on collision. The scattered photon, ω', is emitted at spherical angles (θ, ϕ) with respect to the electron direction, taking the laser beam to lie in the $x - z$ plane. Then following the argument given in Section 5d we find the frequency of the nth harmonic scattering to be

$$\omega' = n\omega \frac{1 + \beta \cos\alpha}{1 - \beta\cos\theta + \Delta}$$

where

$$\Delta = \frac{\eta^2}{2\gamma^2(1 + \beta\cos\alpha)}(1 + \sin\alpha\sin\theta\cos\phi + \cos\alpha\cos\theta).$$

We have neglected a term in $n\omega/\gamma m$ in the above. If $\eta^2 \ll \gamma^2$ then the scattered photons of maximum frequency are emitted at angles $(\theta_{\text{max}}, \phi = -180°)$ where

$$\tan\theta_{\text{max}} \sim \frac{\eta^2 \sin\alpha}{2\gamma^2(1 + \cos\alpha)}$$

supposing $\gamma \gg 1$. The maximum frequency is

$$\omega'_{\text{max}} \sim \frac{2\gamma^2 n\omega(1 + \cos\alpha)}{1 + \eta^2}.$$

The non-zero value of θ_{max} is due to the deflection of the electron beam by the laser beam. Note again the substantial frequency shift for radiation emitted by electrons inside the strong field.

For a laser beam with $\eta \sim 1$ it is appropriate to detect the first four harmonics of the scattered radiation. Say that the $n = 4$ harmonic is desired at 250 nm wavelength, so the fundamental should be placed at 1 μm. For a CO_2 laser the wavelength is 10 μm, so the doppler shift expression tells us that we need

$$\frac{2\gamma^2(1 + \cos\alpha)}{1 + \eta^2} = 10.$$

For a laser beam with $\eta = 1$ and a 5 MeV electron beam ($\gamma = 10$) the appropriate angle for the laser beam is $\alpha = 154°$. That is, the laser beam and the electron beam should converge on each other with 26° between them. In this example the deflection of the electron beam is 1.3°.

The frequency of the scattered light decreases as we move away from $\theta = 0$. To get a sense of this effect is is useful to consider the transformation of the scattered photon from the average rest frame of the electron into the lab. From this we find

$$\frac{\omega'}{\omega'_{\text{max}}} \sim \frac{1 + \cos\theta^\star}{2}$$

and

$$\sin\theta \sim \frac{\sin\theta^\star}{\gamma_{\|}(1 + \cos\theta^\star)} = \frac{\sqrt{1 + \eta^2}\sin\theta^\star}{\gamma(1 + \cos\theta^\star)}$$

again supposing $\gamma \gg 1$. If we collect light over the range $0° \leq \theta^\star \leq 60°$ then we obtain about 25% of the scattered light with a resulting spread in frequencies of 25%. In the laboratory this light appears in a cone of half angle 4.7° about the electron direction, for $\gamma = 10$ and $\eta = 1$.

The electron beam must be deflected by a magnetic field after passing through the laser beam so that the forward scattered light may be detected. In the geometry under discussion we collect the scattered light down to 20° from the laser beam. If the laser beam is not to hit the collection optics it cannot be focussed with a lens stronger than f/d = 1.4.

To obtain a field strength of $\eta \sim 1$ the laser beam must be focussed down to a spot size only a few wavelengths in diameter. The intersection of this spot with the electron beam provides a well localized source of scattered light, very suitable for analysis in a spectrograph such as that used by Carman *et al.* (1981).

The electron beam must be of sufficient quality not to blur the spectrum of the scattered photons, while being intense enough to yield a sizable counting rate. The energy spread of the electron beam should be small compared to the 25% spread in photon frequencies expected from mono-energetic electrons. The angular divergence of the electron beam should be small compared to 5°.

Only those electrons which intersect the focussed laser beam are useful. As an example, suppose the CO_2 laser pulse is 1 n-sec long and has a beam waist of 5 wavelengths, or 50 μm. We estimate that the electron beam could be brought to a 250 μm spot size with only 1° divergence. Then approximately 1/16 of the electrons would be useful. If the electron linac can deliver 500 mA current for 1 n-sec then $\sim 10^9$ electrons cross the laser focus. As noted in Section 5e about 1% of the electrons will scatter photons. Hence we expect a signal of about 10^7 scattered photons per laser pulse, spread over the various harmonics. This should be sufficient to demonstrate the non-linear Thomson effect.

b. Experiments with GeV electrons.

If a high power Nd:glass or CO_2 laser could be brought to a highly relativistic electron beam such as that at SLAC interesting experiments could be performed. The non-linear Thomson effect could be readily demonstrated, with the scattered photons taking on GeV energies. Also, the large Lorentz boost to the electron's rest frame makes the apparent field strength, γE, approach the critical value m^2/e for stimulated pair creation. Likewise $\gamma\omega \sim m$ so that direct observation of light-by-light scattering would be feasible (using a high energy photon beam derived from the electron beam). Finally, with advances in laser technology we anticipate achieving field strengths $\gamma E \sim 10^{17}$ volts/cm which would allow demonstration of the Unruh radiation effect. All of these considerations indicate that the future of experimental studies of non-linear quantum electrodynamics lies in the development of high powered lasers at the Stanford Linear Accelerator Laboratory.

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