Relativity of Steady Energy Flow

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1 Problem

If a continuous medium has a internal stress, either compressive or tensile, it has an internal density of mass/energy larger than that of the same medium when unstressed.

If this medium is in motion, the moving density of stress-induced mass/energy constitutes an apparent flow of energy that is consistent with the $\mathbf{F} \cdot \mathbf{v}$ power associated with the internal forces whose points of application are moving.

In systems involving steady motion, this flow of energy is only apparent to observers for which a stressed part of the system is in motion. In a system with stressed parts that have motion relative to each other, observers identify a flow of energy only in those parts that are in motion relative to the observer. Hence, different observers can have different views as to the path of the steady flow of energy in the system. This contrasts with the case of pulsed energy flow, in which all observers could identify the position of a localized disturbance, and would associate energy transport with that pulse.

The relativity of steady energy flow is a theme of prob. 65 of [1], in which one part of the system is a moving elastic belt. The technical complexity of belt drives, in which necessary slippage at the drive pulleys [2, 3] tends to excite stress waves [4, 5], perhaps obscures the relativity of energy flow in this example [6].

For a possibly simpler example in which the relativity of steady energy flow can be discussed from both macroscopic and microscopic points of view, consider a 1-dimensional ideal gas consists of n point molecules of mass m that are confined within a region of length L. The temperature T can be modeled by supposing that all molecules have the same speed $v = \sqrt{kT/m}$, where k is Boltzmann's constant.

If the confining walls/pistons are fixed, the gas exerts a pressure (= force F for the 1-dimensional case) given by,

$$F = \frac{nkT}{L} = \frac{nmv^2}{L} = \frac{2mv}{2L/nv}\,, \tag{1}$$

which is equal to the momentum change $2mv$ in an elastic collision with the fixed piston divided by the average time $2L/nv$ between such collisions. This force is transmitted from one piston to gas molecules and then to the other piston by the motion of the molecules. The average momentum of the molecules is zero if the pistons are at rest. However, the rate of momentum transfer is $dp/dt = F$, so that the gas can be said to transfer momentum without possessing it [7].

Suppose the left piston moves to the right with a constant velocity $\delta v \ll v$, and the right piston moves to the right with the same small velocity. Then, the left piston does work on the gas at the rate $F\delta v$, and the gas does work on the right piston at the same rate. Discuss the flow of energy and momentum through the gas in this case.

2 Solution

For two other examples the illustrate relativistic issues of steady energy flow, see [8, 9].

If the pistons have speed δv, an elastic collision of a molecule with the left (right) piston lead to an increase (decrease) of the speed of the molecule by $2\,\delta v$. The average speed of the molecules at temperature T is $v = \sqrt{kT/m}$, so in our simple model the right-moving molecules have speed $v + \delta v$ and the left-moving molecules have speed $v - \delta v$. Half of the molecules are left-moving and half are right moving at any moment in time, so the total momentum of the gas molecules is,

$$p_{\rm total} = \frac{n}{2} m(v + \delta v) - \frac{n}{2} m(v - \delta v) = nm\delta v, \tag{2}$$

and the (linear) momentum density in the gas is,

$$\rho_p = \frac{p_{\rm total}}{L} = \frac{nm\delta v}{L}\,. \tag{3}$$

Energy is being added to the gas at the left piston at rate $F\,\delta v = nmv^2\,\delta v/L$, and this energy is continously being removed at the same rate at the right piston. Hence, the flow of energy through the gas is,

$$\frac{dE}{dt} = F\,\delta v = \frac{nmv^2\,\delta v}{L}\,. \tag{4}$$

We verify this result by calculating the flow of energy past, for example, the point midway between the two pistons. The right-moving molecules have kinetic energy $m(v+\delta v)^2/2$ and the number of such molecules passing the midpoint per unit time is their linear number density $n/2L$ times their velocity v (relative to the moving midpoint between the pistons), so the right-moving energy flow is $(nv/2L)m(v + \delta v)^2/2$. Thus, the energy flow through the gas is,[1]

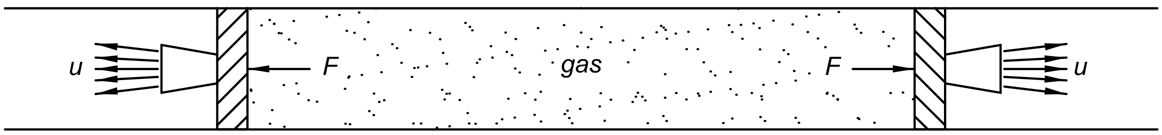
$$\frac{nv}{2L}\frac{m(v+\delta v)^2}{2} - \frac{nv}{2L}\frac{m(v-\delta v)^2}{2} = \frac{nmv^2\,\delta v}{L} = \frac{dE}{dt}\,. \tag{5}$$

An observer who moves to the right with velocity δv sees no motion of the pistons, and so considers that they are doing no work. According to this observer, both the left- and right-moving molecules have speed v, so the energy flow and the momentum density in the gas are both zero. This suggests that the concept of (steady) energy flow is a relative one, as is the point of prob. 65 of [1]. To illustrate this further, we consider two more detailed examples.

[1] We could also consider the total energy flow in the gas, which combines the flow through the gas with the bulk flow of kinetic energy density $nmv^2/2L$ at velocity δv to give a result 3/2 times that of eq. (5).

2.1 The Force on the Pistons is Provided by Rockets

As an idealized 3-d realization of the present problem suppose the gas is confined inside a long, narrow pipe inside which the two pistons slide without friction, as sketched below. There is no tension or compression in the pipe in this case.



Each piston is equipped with a rocket that emits mass at a rate dM/dt at speed u relative to the piston, such that the reaction force on the piston is,

$$F = u\frac{dM}{dt}\,. \tag{6}$$

We suppose that energy is conserved in the rocket engine, so that the internal energy of each piston decreases at the rate,

$$\frac{dE_p}{dt} = \frac{u^2}{2}\frac{dM}{dt}\,, \tag{7}$$

which equals the rate of creation of kinetic energy in the rocket exhaust.

In the rest frame of the pistons, no energy is being transferred from one piston + rocket to the other. The center of mass of the system remains at the midpoint between the pistons in this frame.

But in a frame in which the pistons appear to move to the right with speed δv, the left piston appears to be doing work at the rate $F\,\delta v$, and the right piston has work done on it at the same rate.

The speed of the exhaust of the left rocket appears to be $u - \delta v$ in this frame, while the speed of the exhaust of the right rocket appears to be $u + \delta v$. The rate of creation of kinetic energy by the left rocket is $dM/dt(u - \delta v)^2/2$, while the left piston is losing energy at the rate $dE_p/dt + dM/dt\delta v^2/2$. Altogether, the rate of change of energy of the left piston + rocket + exhaust is,

$$\frac{dE_l}{dt} = \frac{dM}{dt}\left(\frac{(u-\delta v)^2}{2} - \frac{u^2}{2} - \frac{\delta v^2}{2}\right) = -\frac{dM}{dt}u\,\delta v = -F\,\delta v. \tag{8}$$

Similarly, the rate of change of energy of the right piston + rocket + exhaust is,

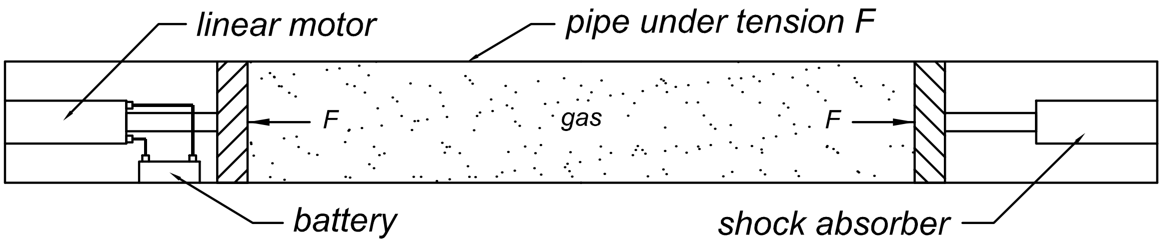
$$\frac{dE_r}{dt} = \frac{dM}{dt}\left(\frac{(u+\delta v)^2}{2} - \frac{u^2}{2} - \frac{\delta v^2}{2}\right) = \frac{dM}{dt}u\,\delta v = F\,\delta v. \tag{9}$$

As noted in eq. (5), the flow of energy in the gas is $F\,\delta v$ in a frame where the pistons move with speed δv, so in this frame it appears that energy is being transferred from the left piston + rocket + exhaust through the gas to the right piston + rocket + exhaust.

In this example, energy appears to be transferred only if the gas appears to have bulk motion. If energy appears to be transferred, it is transferred through the gas because of the difference in speed of the left- and right-moving molecules.

2.2 The Force on the Pistons is Linked to the Pipe that Confines the Gas

An example more representative of an actual pneumatic power transmission system is below. A battery powers a linear motor that exerts force F on the left piston. The right piston is connected to an object that can dissipate energy, such as a shock absorber.



An important distinction between this and the previous example is that the pipe that confines the gas is now under tension F.

Also, if the pistons are moving relative to the right relative to the pipe, then all observers will agree that the battery loses internal energy, and the shock absorber gains internal energy. And then according to the equivalence of mass and energy [10], all observers will agree that mass is being transferred from the battery to the shock absorber.

We now consider the flow of energy from the perspective of observers for which the pipe is at rest, and those for which the pistons are at rest.

2.2.1 The pipe is at rest and the pistons are at rest with respect to the pipe

The battery does no work as the pistons are at rest with respect to the pipe.

The pistons do no work as they appear to be at rest.

There appears to be no flow of energy in this case.

2.2.2 The pipe has speed δv and the pistons are at rest with respect to the pipe

Again, the battery does no work as the pistons are at rest with respect to the pipe.

However, since the system is moving, say, to the right, the pistons do work at the rate $\pm F\,\delta v$, where the $+(-)$ sign applies to the left (right) piston.

The right-moving molecules appear to have velocity $v+\delta v$ while the left-moving molecules appear to have velocity $v-\delta v$. Hence, there appears to be an energy flow to the right through the gas at the rate $F\,\delta v$, as in eq. (5).

Since the battery does no work in this case, we expect that there is no net flow of energy from left to right in the system. Hence, there must appear to be a flow of energy at rate $F\,\delta v$ in the pipe that confines the gas.

Indeed, the pipe is under tension F and is moving to the right with speed δv. Hence, at any cross section of the pipe, the righthand side of the pipe appears to be doing work on the lefthand side of the pipe at rate $F\,\delta v$. This implies an apparent flow of energy from right to left through the pipe at rate $F\,\delta v$, which compensates for the apparent flow of energy from left to right through the gas.

Although there is no net flow of energy in this case, there appears to be a circulation of energy from left to right through the gas, and then from right to left through the pipe.[2]

This circulation of energy is apparent only to observers for whom the system appears to be in motion along the axis of the pipe.

For completeness, we note that this apparent circulation of energy is consistent with the relativistic, (symmetric) mechanical energy-momentum-stress tensor, which can be written as,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = \left(\begin{array}{c|c} u_{\rm mech} & c\boldsymbol{\rho}_{\rm mech} \\ \hline \mathbf{S}_{\rm mech}/c & -T^{ij}_{\rm mech} \end{array} \right), \tag{10}$$

where the mechanical energy density is $u_{\rm mech} = \rho_m c^2$, the mass density is ρ_m, the speed of light is c, the mechanical energy flux (including the flux of rest mass/energy) is $\mathbf{S}_{\rm mech}$, the density of mechanical momentum is related by,

$$\boldsymbol{\rho}_{\rm mech} = \frac{\mathbf{S}_{\rm mech}}{c^2}, \tag{11}$$

and $T^{ij}_{\rm mech}$ is the 3-dimensional mechanical stress tensor.

For example, in the rest frame of a gas at pressure $P = F/A_g$ the energy-momentum-stress tensor is given by,

$$\mathsf{T}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho^\star_g c^2 & 0 & 0 & 0 \\ \hline 0 & F/A_g & 0 & 0 \\ 0 & 0 & F/A_g & 0 \\ 0 & 0 & 0 & F/A_g \end{array} \right), \tag{12}$$

where $\rho^\star_g$ is not simply the rest-mass density of the gas, but includes the relativistic correction to the mass of the moving molecules.

The Lorentz transformation L_z from the rest frame to a frame in which the system has velocity $v\,\hat{\mathbf{z}}$ can be expressed in tensor form as,

$$\mathsf{L}^{\mu\nu}_z = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma\beta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{array} \right), \tag{13}$$

[2]For a discussion of angular momentum in such apparent circulations of energy, see [8].

where $\beta = v/c$. Hence, the energy-momentum-stress tensor in that frame is given by,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = (\mathsf{L}_z \mathsf{T}^\star_{\rm mech} \mathsf{L}_z)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \rho_g^\star c^2 + \gamma^2 \beta^2 F/A_g & 0 & 0 & \gamma^2 \beta(\rho_g^\star c^2 + F/A_g) \\ \hline 0 & F/A_g & 0 & 0 \\ 0 & 0 & F/A_g & 0 \\ \gamma^2 \beta(\rho_g^\star c^2 + F/A_g) & 0 & 0 & \gamma^2 \beta^2 \rho_g^\star c^2 + \gamma^2 F/A_g \end{array} \right). \tag{14}$$

The total mechanical energy flux has only a z component,

$$S_{{\rm mech},z} = \gamma^2 v(\rho_g^\star c^2 + F/A_g) \approx v(\rho_g^\star c^2 + F/A_g), \tag{15}$$

where the approximation holds for $v \ll c$. As expected, the energy flux $S_{{\rm mech},z}$ consists of a the transport of energy/mass density $\rho_g^\star c^2$ at velocity v plus the energy flux Fv/A_g associated with the moving molecules in the gas whose pressure is $P = F/A_g$ in the average rest frame of the gas.

For a pipe under tension F along the z axis, the energy-momentum-stress tensor in the rest frame of the pipe is,

$$\mathsf{T}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho_p^\star c^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -F/A_p \end{array} \right), \tag{16}$$

where A_p is the cross-sectional area of the pipe. In a frame where the pipe has velocity $v\,\hat{\mathbf{z}}$ the stress tensor has the form,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = (\mathsf{L}_z \mathsf{T}^\star_{\rm mech} \mathsf{L}_z)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \rho_p^\star c^2 - \gamma^2 \beta^2 F/A_p & 0 & 0 & \gamma^2 \beta(\rho_p^\star c^2 - F/A_p) \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma^2 \beta(\rho_p^\star c^2 - F/A_p) & 0 & 0 & \gamma^2 \beta^2 \rho_p^\star c^2 - \gamma^2 F/A_p \end{array} \right). \tag{17}$$

The energy flux contains a term $\approx -vF/A_p$ for $v \ll c$, corresponding to an apparent steady flow of energy in the pipe in the opposite direction to the motion of the pipe. No velocity can be associated with this apparent counterpropagating energy flow as we have no microscopic model of the behavior of the pipe under tension.[3]

The total flow of energy associated with the pressure in the gas and the tension in the pipe is $A_g(vF/A_g) + A_p(-vF/A_p) = 0$, as expected.

[3] In a solid where the electrons of neighboring atoms interact slightly, the characteristic velocity of atomic electrons, $\alpha c \approx 0.01c$, may be the relevant velocity of the energy flow.

2.2.3 The pipe is at rest and the pistons have speed δv with respect to the pipe

This is the situation discussed in the text around eqs. (2)-(5). There is a transfer of energy/mass from the battery to the shock absorber, and an observer for which the pipe is at rest identifies the flow of energy as being through the gas.

The pipe is under tension, and so has an energy density greater than that of an unstressed pipe, but there is no energy flow associated with this energy density in a frame where the pipe is at rest.

The energy-momentum-stress tensor (14) alerts us to small, relativistic corrections to eqs. (1)-(5). In the rest frame of the pistons, the force (pressure) is actually,

$$\frac{F}{A} = \frac{\gamma_v n m v^2}{L}, \tag{18}$$

where,

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{19}$$

because the relativistic momentum change in a collision is $2\gamma_v m v$ rather than $2mv$. The mass/energy density in the rest frame is,

$$\rho_m^\star = \frac{\gamma_v n m c^2}{L}. \tag{20}$$

Then, from eq. (15) we find the total energy flux in *the gas to be,*

$$S = \gamma_{\delta v}^2 \frac{\gamma_v n m c^2}{L} \delta v \left(1 + \frac{v^2}{c^2}\right). \tag{21}$$

The first term of eq. (21) is the flux of mass/energy due to the bulk motion of the gas, and the second, smaller term is the flux of energy through *the gas.*

The density of momentum in the gas is,

$$\rho_p = \frac{S}{c^2} = \gamma_{\delta v}^2 \frac{\gamma_v n m}{L} \delta v \left(1 + \frac{v^2}{c^2}\right). \tag{22}$$

The first term of eq. (22) is the momentum density due to the bulk motion of the gas, and the second, smaller term is the momentum density associated with the flux of mass/energy through *the gas.*

To obtain eqs. (21)-(22) in the manner of eqs. (2) and (4) we must take into account the relativistic mass increase, the relativistic velocity transformation, and the relativistic transformation of number density of the left- and right-moving molecules.

2.2.4 The pipe is moving and the pistons are at rest

According to an observer who moves to the right at speed δv, in situation 3 the pistons are at rest, the pipe moves to the left with speed δv, and energy is transferred from the battery to the shock absorber. The energy does not appear to be transmitted through the gas, as both the left- and right-moving molecules have speed v. Rather, the energy appears to be transmitted along (and inside) the wall of the pipe.

The pipe is under tension F so that at any cross section of the pipe the left side appears to be doing work on the right side at rate $F\,\delta v$. The energy appears to flow from the battery, down the (left-moving) pipe to the right, and into the shock absorber.[4]

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Translation: *Does the Inertia of a Body Depend upon its Energy-Content?*, http://kirkmcd.princeton.edu/examples/EM/einstein_ap_18_639_05_english.pdf

[4]To get from the battery the pipe the energy is transmitted for a short distance to the left by the linear motor. The rods of the motor are under compression so the flow of energy in them has the same direction as their bulk motion, *i.e.*, to the left.