Electrodynamics in 1 and 2 Spatial Dimensions

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1 Problem

In theoretical physics, consideration of phenomena in one or two spatial dimensions (1 + 1 or 2 + 1 spacetime dimensions) is often regarded as a useful step toward understanding in three spatial dimensions (3 + 1 spacetime).[1] Why isn't classical electrodynamics more often considered in one or two spatial dimensions?[2,3,4,5,6]

2 Solution

2.1 Electrodynamics in 1 Spatial Dimension

If we consider the case of a single spatial dimension as similar to that of three dimensions but with all charges and field lines somehow confined inside a narrow tube, we are led to suppose that the electric field strength is constant (in both space and time) away from charges, since a tube of field lines has constant field strength. Then, there are no waves of electric field away from charges, even if the charges are moving/accelerating.[7]

[1] A classic example is the Ising model of ferromagnetism [1].

[2] Examples of Schrödinger's equation in one and two spatial dimensions include the case of a " Coulomb" potential $V = -q/r$, but this potential does not correspond to the electric potential of a (point) charge q except in three spatial dimensions [2]. As can be confirmed from sec. 2, the electric scalar potential in one spatial dimension is $V_1 = -q\,|x|$ [3], while in two spatial dimensions it is $V_2 = -q \ln r$ [4].

[3] Aspects of this problem were considered by Ehrenfest [5, 6] as a partial "explanation" as to why we live in 3 + 1 spacetime. See [7] for general comments on such efforts.

[4] While the literature on electrodynamics in integer spatial dimensions other than 3 is sparse, there is a very extensive literature on electrodynamics in fractal dimensions. See, for example, [8, 9, 10, 11].

[5] For general relativity in one and two spatial dimensions, see [12].

[6] The most famous literary excursion into two spatial dimensions is *Flatland* [13] (1884). Also noteworthy is *The Planiverse* by Dewdney [14, 15], which inspired considerations of electrodynamics in two spatial dimensions in [16, 17].

[7] Different versions of electrodynamics was discussed by Wheeler, sec. 8 of [18], and [19], using the language of differential forms to construct electrodynamics in an odd number of spatial dimensions, but not in an even number, in a manner that the case of 3 spatial dimensions is the familiar Maxwellian electrodynamics. [Feb. 21 (2022)] See also sec. IV.A.2 of [20].

This version of electrodynamics in 1 spatial dimension has both a scalar E and B, field, but no electric charge (and so is not a gauge theory). Rather, the fields are coupled to a scalar current J, and obey,

$$\frac{\partial B}{\partial x} = \frac{1}{c}\frac{\partial E}{\partial t} + J, \qquad \frac{\partial E}{\partial x} = \frac{1}{c}\frac{\partial B}{\partial t}, \tag{1}$$

This electrodynamics does support waves, according to,

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial E}{\partial t^2} = -\frac{1}{c}\frac{\partial J}{\partial t}\,, \qquad \frac{\partial^2 B}{\partial x^2} - \frac{1}{c^2}\frac{\partial B}{\partial t^2} = -\frac{\partial J}{\partial x}\,. \tag{2}$$

Furthermore, the magnetic field (in 3 spatial dimensions) of moving charges does not affect other charges moving along the same axis. In effect, there is no magnetic field in 1 + 1 electrodynamics as considered here (*i.e.*, in 1 spatial dimension, with only electric charges). As there are no waves and no magnetic field in 1 spatial dimension, the speed of light, c, plays no role in "electrodynamics" here.[8]

It is natural to choose (Gaussian) units in which the (Coulomb) force between two charges q and q', which is independent of their separation, is simply $F = qq'$, with the implication that the magnitude of the electric field strength E of charge q is just $E = q$ for observers at $x > x_q$. Then, the electric field strength at a point x_0 is $E = q_+ - q_-$, where q_+ is the total charge at $x > x_0$ and q_- is the total charge at $x < x_0$, independent of the motion of these charges (unless they cross the point x_0 so as to change q_+ and q_-).

There is not much more content to classical electrodynamics in 1 + 1 dimensions, which seems too trivial to provide useful "toy models" to help in understanding electrodynamics in 3 + 1 dimensions.[9,10,11]

2.1.1 Additional Remarks

This section added Aug. 2019, inspired by e-comments from Xabier Prado and Jorge Mira. A slightly different perspective on electrodynamics in 1 spatial dimension is given in Appendix A.1

A differential equation for the 1-dimensional electric field can be written as,

$$\frac{\partial E}{\partial x} = 2\rho, \tag{3}$$

where ρ is the (linear) charge density. This can be considered the 1-d version of the 3-d Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ (in Gaussian units). According to eq. (3), the electric field of charge q at position x_q is $E = q$ for $x > x_q$ and $E = -q$ for $x < x_q$. The total electric field at position x is,

$$E(x) = Q_- - Q_+, \tag{4}$$

where $Q_+ = \int_x^\infty \rho(x')\,dx'$ is the total charge at and $x' > x$ and $Q_- = \int_{-\infty}^x \rho(x')\,dx'$ is the total charge at and $x' < x$.

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[8] Electromagnetic plane waves in three spatial dimensions are often considered to be "one dimensional", but such waves don't exist in "one dimensional" (1 + 1 spacetime) electrodynamics. For an example of how one-dimensional thinking can lead to misunderstandings as to three-dimensional electrodynamics, see [24].

[9] An electric dipole, with charge q at x_+ and charge $-q$ at x_- has zero electric field outside the interval $[x_+, x_-]$, so there is no interaction between and one electric dipole and another, or between an electric dipole and an external electric charge. Thus, 1-dimensional models of spin-spin interactions are outside the type of 1-dimensional electrodynamics considered here.

[10] Schwinger's study [25] of quantum electrodynamics in 1 + 1 dimensions anticipated, to some extent, the Higgs mechanism in the emerging electroweak theory.

[11] The theory of quantum chromodynamics includes the phenomenon of "confinement" of the color fields, which can be approximated as lying within 1-dimensional flux tubes/strings. Hence, toy models of 1 + 1 dimensional quantum chromodynamics can provide some insight into the 3 + 1 dimensional case [26, 27, 28].

There is no need for an additional differential equation to determine the scalar field E, but we note that the 3-d Maxwell equation $\boldsymbol{\nabla}\times\mathbf{B} = (1/c)\partial\mathbf{E}/\partial t + 4\pi\mathbf{J}/c$ has a 1-d equivalent,

$$0 = \frac{\partial E}{\partial t} + 2J, \tag{5}$$

where J is the 1-d current density. We could call $\partial E/\partial t$ the 1-d "displacement current", in which case eq. (5) could be interpreted as the total 1-d current being zero (although such language does not add much to our understanding).

Equations (3) and (5) are not independent, as seen by taking derivatives,

$$\frac{\partial^2 E}{\partial t\partial x} = 2\frac{\partial\rho}{\partial t} = \frac{\partial^2 E}{\partial x\partial t} = -2\frac{\partial J}{\partial x}\,, \qquad \frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0, \tag{6}$$

which latter is the 1-d continuity equation (that expresses conservation of charge).

In a region where the charge and current densities are zero, the electric field is constant in both x and t, so there are no (nontrivial) electric waves. The electric field at a point can only change if the charge or current density at the point changes.

There is no stability for an electric charge in a region where the only external force is due to the electric field of other charges. Thus, the 1-d version of Earnshaw's theorem [21] is stronger than that in 3-d.

Since electrodynamics in one spatial dimension is so trivial, there is little advantage to consideration of potentials. But, we can contemplate both a scalar potential V and a "vector" potential A such that,[12]

$$E = -\frac{\partial V}{\partial x} - \frac{\partial A}{\partial t}\,. \tag{7}$$

Since there is only a single, scalar field E, V and A cannot be independent, so they are subject to a (gauge) condition. Examples are:

1. $A = 0$. This condition can only apply to 1-d electrodynamics, where $\mathbf{B}\ (= \boldsymbol{\nabla}\times\mathbf{A}) = 0$.

   In this gauge, $V(x) = \int \rho(x')\,|x' - x|\ dx'$.

2. $V = 0$, which condition defines the so-called Gibbs (or Hamiltonian) gauge [23].

   In this gauge, $A(x,t) = -\int_{t_0}^{t} E(x,t')\,dt'$.

3. $\partial A/\partial x = -(1/v^2)\,\partial V/\partial t$, which condition defines a velocity gauge [23] for arbitrary (constant) velocity v. The Coulomb gauge corresponds to $v = \infty$, while $v = c$ defines the Lorenz gauge.

   Combining eqs. (5), (7) and the velocity-gauge condition, we find a wave equation for A:

$$-2J = \frac{\partial E}{\partial t} = -\frac{\partial^2 V}{\partial t\partial x} - \frac{\partial^2 A}{\partial t^2} = v^2\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial t^2}\,, \qquad \frac{\partial^2 A}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 A}{\partial t^2} = -\frac{2J}{v^2}\,. \tag{8}$$

[12]See, for example, sec. V of [22].

It is impressive that there can be waves of the "vector potential" A, since there are no waves of the field E. However, there is not much physical content to eq. (8), as can be seen by the following.

In a region where J is zero, a wavefunction for A has the form $A = f(x - vt)$. The gauge condition tells that $f' = -(1/v^2)\, \partial V/\partial t$, such that $V = v f(x - vt)$, and then eq. (7) leads to $E = -vf' + vf' = 0$, as expected since there are no waves of E in 1-d electrodynamics.

2.2 Electrodynamics in 2 Spatial Dimensions

In two spatial dimensions, Coulomb's law for the force on electric charge q' due to charge q, both at rest, is $\mathbf{F}_{q'} = qq'\,\hat{\mathbf{r}}/r \equiv q'\,\mathbf{E}$, where $\mathbf{r}$ is the vector from charge q to q' and we use Gaussian units, so the electric field $\mathbf{E}$ of a point charge q (at rest) has the nontrivial form,[13]

$$\mathbf{E} = \frac{q}{r}\,\hat{\mathbf{r}}. \tag{9}$$

Moving charges lead to time-varying electric fields at a fixed observer, so wave phenomena are possible.

Gauss' law for the electric field $\mathbf{E} = (E_x, E_y)$ in 2 spatial dimensions is that the number of field lines crossing a closed loop is proportional to the total charge inside the loop,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 2\pi Q_{\rm in} = 2\pi \int \rho\, d\text{Area}, \tag{10}$$

where ρ is the (surface) charge density, and the proportionality constant is 2π in Gaussian units. The differential form of Gauss' law in 2 spatial dimensions is,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 2\pi\rho, \qquad \text{where} \qquad \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right). \tag{11}$$

When both charges q and q' are in motion with uniform velocities $\mathbf{v} = (v_x, v_y)$ and $\mathbf{v}' = (v'_x, v'_y)$ that are small compared to c, charge q' experiences the (Lorentz) force,[14]

$$\mathbf{F}_{q'} = q'\left(\mathbf{E} + (v'_y, -v'_x)\frac{q(v_x r_y - v_y r_x)}{c^2 r^2}\right), \tag{13}$$

[13]Electrodynamics in two spatial dimensions (x, y) is equivalent to that in three spatial dimensions (x, y, z) for examples in which the 3-d charge and current distributions are independent of z. Since the 3-d electric field of a line charge of density λ per unit length is $\mathbf{E}_3 = 2\lambda\,\hat{\mathbf{r}}/r$, where $\mathbf{r}$ lies in the x-y plane, we see that a 2-d electric charge q is equivalent to a 3-d linear charge density $\lambda = q/2$.

The 3-d electrostatic force between line charges λ and λ' is $2\lambda'\lambda\,\hat{\mathbf{r}}/r$, which corresponds to $q'\mathbf{E}_3/2$. Thus, the 2-d force must be twice the corresponding 3-d force, if we wish to have the relation $\mathbf{F} = q'\mathbf{E}_2$ hold in 2-d.

[14]The 2-d magnetic force corresponds to the 3-d force per unit length in 3-d examples where the charge and current densities are independent of z. Since the 3-d magnetic field of a slowly moving line charge of density $\lambda = q/2$ with velocity $\mathbf{v}$ is $\mathbf{v}/c \times \mathbf{E}_3 = \mathbf{v}/c \times q\,\mathbf{r}/r^2$, the 3-d magnetic force per unit length on a line charge of density $\lambda' = q'/2$ with velocity $\mathbf{v}'$ is, where vectors $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{v}'$ have no z component,

$$\begin{aligned} \frac{q'q}{2c^2r^2}\mathbf{v}' \times (\mathbf{v} \times \mathbf{r}) &= \frac{q'q}{2c^2r^2}[(\mathbf{v}' \cdot \mathbf{r})\mathbf{v} - (\mathbf{v} \cdot \mathbf{v}')\,\mathbf{r}] = \frac{q'q}{2c^2r^2}[(v'_x r_x + v'_y r_y)(v_x, v_y, 0) - (v_x v'_x + v_y v'_y)(r_x, r_y, 0)] \\ &= \frac{q'q}{2c^2r^2}(v_x r_y - v_y r_x)(v'_y, -v'_x, 0), \end{aligned} \tag{12}$$

from which eq. (13) follows, noting that the 2-d force is twice the corresponding 3-d force.

where again vector $\mathbf{r}$ points from q to q'. This might lead enthusiasts of vector analysis to invent the concept of "perpendicular vectors", defined for any 2-vector $\mathbf{a} = (a_x, a_y)$ as,[15]

$$\mathbf{a}_\perp \equiv (a_y, -a_x), \qquad (\mathbf{a}_\perp)_\perp = -\mathbf{a}, \tag{14}$$

so that,[16]

$$\mathbf{a}_\perp \cdot \mathbf{a}_\perp = \mathbf{a} \cdot \mathbf{a} = a^2 = a_x^2 + a_y^2, \qquad \mathbf{a} \cdot \mathbf{a}_\perp = 0, \qquad \mathbf{a} \cdot \mathbf{b}_\perp = -\mathbf{a}_\perp \cdot \mathbf{b} = a_x b_y - a_y b_x. \tag{15}$$

Then, the Lorentz force law can be recast as,[17]

$$\mathbf{F}_{q'} = q' \left(\mathbf{E} + \frac{\mathbf{v}'_\perp}{c} B \right), \tag{16}$$

where the scalar B is the "magnetic" field in 2 spatial dimensions due to charge q moving with velocity $\mathbf{v}$ small compared to c,

$$B = q \frac{\mathbf{v} \cdot \mathbf{r}_\perp}{cr^2} \ \left(= -q \frac{\mathbf{v}_\perp \cdot \mathbf{r}}{cr^2} \right). \tag{17}$$

For a continuous, steady (surface) current density $\mathbf{J}$ of moving charges, the magnetic field has the (Biot-Savart) form,

$$B = \int \frac{\mathbf{J} \cdot \mathbf{r}_\perp}{cr^2} \, d\text{Area}, \tag{18}$$

where "steady" means,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = 0. \tag{19}$$

The Ampère of 2 + 1 spacetime might then introduce the 2-dimensional vector derivative operator,

$$\boldsymbol{\nabla}_\perp = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x} \right), \qquad \nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\nabla}_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}_\perp = 0, \tag{20}$$

to show that the scalar magnetic field (18) due to steady currents obeys the differential equation,[18]

$$\boldsymbol{\nabla}_\perp B = \frac{2\pi}{c} \mathbf{J}. \tag{21}$$

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[15]The vector $-\mathbf{a}_\perp$ is also perpendicular to $\mathbf{a}$, so the notion of a perpendicular vector has a two-fold ambiguity in 2 spatial dimensions. In $n \geq 3$ spatial dimensions there is an infinite set of perpendicular vectors (of same norm as an n-vector $\mathbf{a}$), such that perpendicular-vector algebra is not useful there.

[16]$\mathbf{a} \cdot \mathbf{b}_\perp = -\mathbf{a}_\perp \cdot \mathbf{b}$ is the 2-vector equivalent of the third component of $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a}$ and $\mathbf{b}$ are 3-vectors. Thus, perpendicular-vector algebra plays a special role in 2 spatial dimensions, akin to the special role of vector cross products in 3 spatial dimensions.

[17]It does not make sense to suppose that there could exist magnetic charges p in 2 + 1 spacetime, as no 2-vector force $\mathbf{F}_p$ can be constructed from p and B. As discussed in Appendix A, it seems that magnetic charges are physically plausible only in 3 + 1 spacetime.

[18]Ampère had no concept of the magnetic field, and did not invent the differential form, $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi \mathbf{J}/c$, of "Ampère's law" in 3-d, which rather was first stated by Maxwell on p. 56 of [29].

The Faraday of 2 + 1 spacetime would show that a time-varying magnetic field affects the electric field according to the forms,[19]

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt}\int B\, d\text{Area}, \qquad \boldsymbol{\nabla}\cdot\mathbf{E}_\perp = -\boldsymbol{\nabla}_\perp\cdot\mathbf{E} = -\frac{1}{c}\frac{\partial B}{\partial t}\,, \tag{22}$$

and finally the Maxwell of 2 + 1 spacetime would generalize Ampère's law (21) to time-varying situations by inventing the 2 + 1 "displacement current", $(1/2\pi)\partial\mathbf{E}/\partial t$, such that the microscopic Maxwell equations (for $\mathbf{E}$ rather than $\mathbf{E}_\perp$) are,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 2\pi\rho, \qquad \boldsymbol{\nabla}_\perp\cdot\mathbf{E} = \frac{1}{c}\frac{\partial B}{\partial t}\,, \qquad \boldsymbol{\nabla}_\perp B = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{2\pi}{c}\mathbf{J}. \tag{23}$$

There are three scalar, differential equations for the three field components E_1, E_2 and B; there is no equivalent to the 3-dimensional equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.[20]

Applying the operator $\boldsymbol{\nabla}_\perp$ to the third of eq. (23), and then using the second of these, leads to the wave equation for the magnetic field,

$$\nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = \frac{2\pi}{c}\boldsymbol{\nabla}_\perp\cdot\mathbf{J}. \tag{24}$$

Noting a vector calculus identity, $\boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}_\perp\cdot\mathbf{E}) = \nabla^2\mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})$, applying the operator $\boldsymbol{\nabla}_\perp$ to the second of eq. (23), and then using the first and third of these, leads to the wave equation for the electric field,

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 2\pi\boldsymbol{\nabla}\rho + \frac{2\pi}{c^2}\frac{\partial\mathbf{J}}{\partial t}\,. \tag{25}$$

Thus, waves of both $\mathbf{E}$ and B propagate at the speed c, which can be called the speed of "light" in 2 + 1 electrodynamics.

2.2.1 Potentials

To write the fields $\mathbf{E}$ and B in terms of potentials, we don't use a single scalar potential for B, but relate it to a two-component vector potential $\mathbf{A}$, according to,

$$B = -\boldsymbol{\nabla}\cdot\mathbf{A}_\perp = -\boldsymbol{\nabla}_\perp\cdot\mathbf{A}, \tag{26}$$

so that the second Maxwell equation of (23) can be written as,

$$\boldsymbol{\nabla}_\perp\cdot\left(\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = 0. \tag{27}$$

[19]The integral form of Faraday's law in 3-d was first given by Maxwell on p. 50 of [29], and the differential form was first stated by him on p. 290 of [30].

[20]In 3 + 1 spacetime the equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ indicates that the 3-vector magnetic field $\mathbf{B}$ is not due to magnetic charges. In 2 + 1 spacetime the scalar character of the magnetic field B alerts us that this is not due to magnetic charges, which would led to a 2-vector magnetic field $\mathbf{B}_m$. The Appendix considers electrodynamics with magnetic charges in 2 + 1 spacetime.

From the definitions (20) we see that $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_\perp = 0$, so that any vector field $\mathbf{f}$ of the form $-\boldsymbol{\nabla}V$ for a scalar potential V obeys $\boldsymbol{\nabla}_\perp\cdot\mathbf{f} = 0$. Hence we can relate the field $\mathbf{f} = \mathbf{E} + \partial\mathbf{A}/\partial ct$ to a scalar potential V, and write,

$$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\,. \tag{28}$$

Wave equations for the potentials can be found by replacing the fields in the Maxwell equations (23) by their expressions (26) and (28) in terms of the potentials. The first of eq. (23) leads to,

$$\nabla^2 V + \frac{1}{c}\frac{\partial\boldsymbol{\nabla}\cdot\mathbf{A}}{\partial t} = -2\pi\rho, \tag{29}$$

and the third of eq. (23) implies that,

$$\nabla^2\mathbf{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \frac{1}{c}\boldsymbol{\nabla}\frac{\partial V}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{2\pi}{c}\mathbf{J}. \tag{30}$$

If we invoke the Lorenz-gauge condition,

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0 \qquad \text{(Lorenz)}, \tag{31}$$

then the wave equations of the potentials take the forms,

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -2\pi\rho, \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{2\pi}{c}\mathbf{J} \qquad \text{(Lorenz gauge)}. \tag{32}$$

Since cylindrical waves vary with $r_2 = |\mathbf{x}_2 - \mathbf{x}_2'|$ as $1/\sqrt{r_2}$, where $\mathbf{x}_2 = (x, y)$ is the 2-dimensional position vector, one might expect that the retarded-potential solutions to the 2-d wave equations (32) have the form,[21]

$$V_2(\mathbf{x}_2, t) = \int \frac{\rho_2(\mathbf{x}_2', t' = t - r_2/c)}{\sqrt{r_2}}\, d^2\mathbf{x}_2', \qquad \mathbf{A}_2(\mathbf{x}_2, t) = \int \frac{\mathbf{J}_2(\mathbf{x}_2', t' = t - r_2/c)}{c\sqrt{r_2}}\, d^2\mathbf{x}_2'. \tag{33}$$

Now, for a 3-dimensional example in which the charge and current distributions were independent of z (*i.e.*, two dimensional), the retarded scalar potential in the plane $z = 0$ could be written as,

$$\begin{aligned} V_3(\mathbf{x}_3, t) = V_3(\mathbf{x}_2, t) &= \int \frac{\rho_3(\mathbf{x}_3', t' = t - r_3/c)}{r_3}\, d^3\mathbf{x}_3' = 2\int_0^\infty dz' \int \frac{\rho_3(\mathbf{x}_2', t' = t - r_3/c)}{\sqrt{r_2^2 + z'^2}}\, d^2\mathbf{x}_2' \\ &= 2\int_{-\infty}^{t - r_2/c} dt' \int \frac{\rho_3(\mathbf{x}_2', t')}{\sqrt{(t - t')^2 - r_2^2/c_2}}\, d^2\mathbf{x}_2', \end{aligned} \tag{34}$$

where $\mathbf{x}_3 = (x, y, 0)$, $\mathbf{x}_2 = (x, y)$, $r_2^2 = (x - x')^2 + (y - y')^2$, $r_3^2 = r_2^2 + z'^2$, and the relation $t' = t - r_3/c$ leads to $|z'| = c\sqrt{(t - t')^2 - r_2^2/c^2}$ and $dz' = c\, r_3\, dt'/\,|z'|$.

[21] An earlier version of eq. (33) was also faulty, as pointed out in [35], which brought refs. [31, 32, 34] to the author's attention.

Hadamard [31, 32] was among the first to realize that the retarded (scalar) potential V_2 in 2-dimensional electrodynamics is not given by eq. (33), but actually obeys the form (34)[22] (with the 2-d charge density $\rho_2 = 2\rho_3$, as in footnote 10), which contains contributions to the integrand from times earlier than $t' = t - r_2/c$, as if the 2-dimensional space (x, y) had an effective third dimension z. This peculiar behavior (called "descent" from 3 to 2 dimensions by Hadamard)[23] led Ehrenfest (1917) [5, 6] to argue that this is why we live in a 3-dimensional space.[24]

Examples of electrodynamics in 2 + 1 dimensions may be best approached by first considering their equivalents in 3 + 1 dimensions. Hence, there is almost no literature on classical electrodynamics in 2 + 1 dimensions as being simpler than in 3 + 1 dimensions.

2.2.2 Energy, Momentum and Stress

Poynting's argument [37] (originally in 3 + 1 spacetime, but here considered for 2 + 1 spacetime) relates the rate of work done by electromagnetic fields on "free" electric and magnetic currents to both flow of energy and to rate of change of stored energy. The density of the time rate of change of work on electric currents follows from eq. (16),

$$\frac{dw}{dt}\ (= \mathbf{F}\cdot\mathbf{v}) \ = \mathbf{J}\cdot\mathbf{E} = \mathbf{E}\cdot\left(\boldsymbol{\nabla}_\perp B - \frac{1}{2\pi}\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{c}{2\pi}\mathbf{E}\cdot\boldsymbol{\nabla}_\perp B - \frac{1}{4\pi}\frac{\partial E^2}{\partial t}\,. \tag{35}$$

Now,

$$\begin{aligned} \mathbf{E}\cdot\boldsymbol{\nabla}_\perp B &= E_x\frac{\partial B}{\partial y} - E_y\frac{\partial B}{\partial x} = -\frac{\partial(-E_x B)}{\partial y} - B\frac{\partial E_x}{\partial y} - \frac{\partial(E_y B)}{\partial x} + B\frac{\partial E_y}{\partial x} \\ &= -\boldsymbol{\nabla}\cdot\mathbf{E}_\perp B - B\boldsymbol{\nabla}_\perp\cdot\mathbf{E} = -\boldsymbol{\nabla}\cdot\mathbf{E}_\perp B - \frac{B}{c}\frac{\partial B}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{E}_\perp B - \frac{1}{2c}\frac{\partial B^2}{\partial t}\,, \end{aligned} \tag{36}$$

so that eq. (35) can be written as,

$$-\mathbf{J}\cdot\mathbf{E} = \boldsymbol{\nabla}\cdot\left(\frac{c}{2\pi}\mathbf{E}_\perp B\right) + \frac{\partial}{\partial t}\frac{E^2+B^2}{4\pi} \equiv \boldsymbol{\nabla}\cdot\mathbf{S} + \frac{\partial u_{\mathrm{EM}}}{\partial t}\,, \tag{37}$$

where,

$$\mathbf{S} = \frac{c}{2\pi}\mathbf{E}_\perp B, \qquad u_{\mathrm{EM}} = \frac{E^2+B^2}{4\pi}\,. \tag{38}$$

The Poynting vector **S** describes the flux of energy density in the electromagnetic field, and u_{EM} is the density of energy stored in it.

[22] Hadamard's discussion was lengthy, with a somewhat opaque summary in sec. 69, p. 105 of [32]. More compact arguments are given in [33, 34, 35]. See eqs. (51)-(55) of [35] to recover the 2-d static potential of charge q from eq. (34).

[23] For an example, see Appendix B below. (Feb. 17, 2022) For additional comments on Hadamard's method of descent, see [20].

[24] The theme of radiation, and the radiation reaction, in other than three spatial dimensions is the subject of ongoing discussion, as recently reviewed in [36].

Related arguments (first given in 3 + 1 spacetime by Abraham [38] and Minkowski [39]) start from the Lorentz force density as the time rate of change of mechanical momentum,

$$\frac{d\,\mathbf{p}_{\rm mech}}{dt} = \mathbf{f} = \rho\mathbf{E} + \frac{\mathbf{J}_\perp}{c}B = \frac{\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})}{2\pi} + \frac{\mathbf{J}_\perp}{c}B. \tag{39}$$

We can form the perpendicular-vector version of the third Maxwell equation of (23), finding,

$$(\boldsymbol{\nabla}_\perp B)_\perp = -\boldsymbol{\nabla} B = \frac{1}{c}\frac{\partial\mathbf{E}_\perp}{\partial t} + \frac{2\pi}{c}\mathbf{J}_\perp, \tag{40}$$

$$\begin{aligned}\frac{\mathbf{J}_\perp}{c}B &= -\frac{B\boldsymbol{\nabla} B}{2\pi} - \frac{B}{2\pi c}\frac{\partial\mathbf{E}_\perp}{\partial t} = -\frac{\partial}{\partial t}\frac{\mathbf{E}_\perp B}{2\pi c} + \frac{\mathbf{E}_\perp}{2\pi c}\frac{\partial B}{\partial t} - \frac{B\boldsymbol{\nabla} B}{2\pi} \\ &= -\frac{\partial}{\partial t}\frac{\mathbf{E}_\perp B}{2\pi c} - \frac{\mathbf{E}_\perp(\boldsymbol{\nabla}\cdot\mathbf{E}_\perp)}{2\pi} - \frac{\boldsymbol{\nabla} B^2}{4\pi}\,,\end{aligned} \tag{41}$$

such that eq. (39) can be written as,

$$\frac{d\mathbf{p}_{\rm mech}}{dt} = -\frac{\partial}{\partial t}\frac{\mathbf{E}_\perp B}{2\pi c} + \frac{\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})}{2\pi} - \frac{\mathbf{E}_\perp(\boldsymbol{\nabla}\cdot\mathbf{E}_\perp)}{2\pi} - \frac{\boldsymbol{\nabla} B^2}{4\pi} = -\frac{d\mathbf{p}_{\rm EM}}{dt} + \boldsymbol{\nabla}\cdot\mathsf{T}, \tag{42}$$

where,[25]

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E}_\perp B}{2\pi c} = \frac{\mathbf{S}}{c^2} \tag{43}$$

is the density of momentum stored in the electromagnetic field, and T is the 2×2 Maxwell stress tensor,

$$\mathsf{T}_{ij} = \frac{E_i E_j}{2\pi} - \delta_{ij}\frac{E^2+B^2}{4\pi}\,. \tag{44}$$

A Appendix: Microscopic Electrodynamics via $n+1$ Spacetime Vectors

To generalize 3 + 1 spacetime electrodynamics to $n+1$ dimensions, it is useful cast 3 + 1 electrodynamics into (3 + 1)-vector notation. We also consider the possibility of magnetic charges in addition to electric charges.

When Heaviside first presented Maxwell's equations in vector notation [42] he assumed that in addition to electric charge and current densities, ρ_e and $\mathbf{J}_e$, there existed magnetic charge and current densities, ρ_m and $\mathbf{J}_m$, although there remains no experimental evidence for the latter.[26] Maxwell's equations for microscopic electrodynamics are then (in Gaussian units),[27]

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho_e, \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_m, \quad -c\boldsymbol{\nabla}\times\mathbf{E} = \frac{\partial\mathbf{B}}{\partial t} + 4\pi\mathbf{J}_m, \quad c\boldsymbol{\nabla}\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + 4\pi\mathbf{J}_e, \tag{45}$$

[25]The relation (43) between field momentum density and the Poynting vector was deduced earlier by different arguments by Thomson, p. 9 of [40], and by Poincaré [41].

[26]Heaviside seems to have regarded magnetic charges as "fictitious", as indicated on p. 25 of [43].

[27]See [44] for Maxwell's equations in SI units, including a discussion in footnote 7 there about the ambiguous placement of the permeability μ_0 for terms involving magnetic charges.

where c is the speed of light in vacuum.

The factors of 4π in eq. (45) for 3 + 1 electrodynamics are associated with the surface area of a unit 3-sphere. We anticipate that in $n+1$ spacetime, these factors become,

$$S_n = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}\,, \tag{46}$$

such that $S_1 = 2$, $S_2 = 2\pi$, $S_3 = 4\pi$, $S_4 = 2\pi^2$, ...[28]

The fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from potentials according to,

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m, \qquad \mathbf{B} = \mathbf{B}_e + \mathbf{B}_m, \tag{47}$$

$$\mathbf{E}_e = -\boldsymbol{\nabla} V_e - \frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t}\,, \qquad \mathbf{B}_e = \boldsymbol{\nabla}\times\mathbf{A}_e, \tag{48}$$

$$\mathbf{B}_m = -\boldsymbol{\nabla} V_m - \frac{1}{c}\frac{\partial \mathbf{A}_m}{\partial t}\,, \qquad \mathbf{E}_m = -\boldsymbol{\nabla}\times\mathbf{A}_m, \tag{49}$$

where in the Lorenz gauge,

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0, \tag{50}$$

and the (retarded) potentials for odd n are related to the source charge/current densities by,[29]

$$V_{e,m}(\mathbf{x},t) = \int \frac{\rho_{e,m}(\mathbf{x}', t' = t - r/c)}{r}\, d^3\mathbf{x}', \qquad \mathbf{A}_{e,m}(\mathbf{x},t) = \int \frac{\mathbf{J}_{e,m}(\mathbf{x}', t' = t - r/c)}{c\,r}\, d^3\mathbf{x}', \tag{51}$$

with $r = |\mathbf{x} - \mathbf{x}'|$.

The fields $\mathbf{E}$ and $\mathbf{B}$ obey the duality relations that,

$$(c\rho_e, \mathbf{J}_e) \to (c\rho_m, \mathbf{J}_m), \qquad (c\rho_m, \mathbf{J}_m) \to -(c\rho_e, \mathbf{J}_e) \qquad \Rightarrow \qquad \mathbf{E}\to\mathbf{B}, \qquad \mathbf{B}\to -\mathbf{E}. \tag{52}$$

Expressions (45)-(52) can be taken over to the case of n spatial dimensions, except for eqs. (45), (48)-(49) which involve the vector cross product, which is defined only for three spatial dimensions. Hence, we recast these equations into $(n+1)$-dimensional vector/tensor form where a charge-current vector is written $J_\mu = (c\rho, \mathbf{J})$, $\mu = 0, 1, \ldots, n$, a potential is written $A_\mu = (V, \mathbf{A})$, the derivative operator is written $\partial_\mu = (\partial/\partial ct, -\partial/\partial\mathbf{x})$, and the scalar product of two vectors is written $a_\mu b_\mu = a_0 b_0 - \mathbf{a}\cdot\mathbf{b}$.

Conservation of electric and magnetic charge can now be expressed as,

$$\partial_\mu J_\mu = 0 \ \left(= \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}\right). \tag{53}$$

We introduce the antisymmetric field tensors F_e and F_m with components,

$$\mathsf{F}_{e,\mu\nu} = \partial_\mu A_{e,\nu} - \partial_\nu A_{e,\mu}, \qquad \mathsf{F}_{m,\mu\nu} = \partial_\mu A_{m,\nu} - \partial_\nu A_{m,\mu}, \tag{54}$$

[28]While many people prefer to use units such that the factors of 4π do not appear in Maxwell's equations in 3 + 1 electrodynamics, these units obscure the generalization to $n+1$ spacetime.

[29]$\mathbf{A}_e$ is often called the magnetic vector potential, but as its source is the electrical current $\mathbf{J}_e$, it is better called the electric vector potential.

In 3 spatial dimensions (3 + 1 electrodynamics) these tensors have components,

$$\mathsf{F}_e = \begin{pmatrix} 0 & -E_{e,1} & -E_{e,2} & -E_{e,3} \\ E_{e,1} & 0 & -B_{e,3} & B_{e,2} \\ E_{e,2} & B_{e,3} & 0 & -B_{e,1} \\ E_{e,3} & -B_{e,2} & B_{e,1} & 0 \end{pmatrix}, \quad \mathsf{F}_m = \begin{pmatrix} 0 & -B_{m,1} & -B_{m,2} & -B_{m,3} \\ B_{m,1} & 0 & E_{m,3} & -E_{m,2} \\ B_{m,2} & -E_{m,3} & 0 & E_{m,1} \\ B_{m,3} & E_{m,2} & -E_{m,1} & 0 \end{pmatrix}. \quad (55)$$

Then, the tensor relations, where S_n is defined in eq. (46),

$$\partial_\mu \mathsf{F}_{e,\mu\nu} = \frac{S_n}{c} J_{e,\mu}\,, \qquad \partial_\mu \mathsf{F}_{m,\mu\nu} = \frac{S_n}{c} J_{m,\mu}, \tag{56}$$

lead to the Maxwell equations in $n = 3$ spatial dimensions,

$$\boldsymbol{\nabla} \cdot \mathbf{E}_e = 4\pi\rho_e, \quad c\boldsymbol{\nabla} \times \mathbf{B}_e = \frac{\partial E_e}{\partial t} + 4\pi \mathbf{J}_e, \quad \boldsymbol{\nabla} \cdot \mathbf{B}_m = 4\pi\rho_m, \quad -c\boldsymbol{\nabla} \times \mathbf{E}_m = -\frac{\partial B_m}{\partial t} + 4\pi \mathbf{J}_m. \tag{57}$$

The remaining Maxwell equations,

$$\boldsymbol{\nabla} \cdot \mathbf{B}_e = 0, \qquad -c\boldsymbol{\nabla} \times \mathbf{E}_e = \frac{\partial B_e}{\partial t}\,, \qquad \boldsymbol{\nabla} \cdot \mathbf{E}_m = 0, \qquad c\boldsymbol{\nabla} \times \mathbf{B}_m = \frac{\partial E_m}{\partial t}\,, \tag{58}$$

can be obtained from,

$$\partial_\lambda F_{e,\mu\nu} + \partial_\mu F_{e,\nu\lambda} + \partial_\nu F_{e,\lambda\mu} = 0, \qquad \partial_\lambda F_{m,\mu\nu} + \partial_\mu F_{m,\nu\lambda} + \partial_\nu F_{m,\lambda\mu} = 0, \tag{59}$$

which are true for any values of the indices $\{\lambda, \mu, \nu\}$, but which are nontrivial only if all three indices are distinct, leading to only four different relations for indices $\{1, 2, 3\}$ (which corresponds to the divergence equations in (58)), and $\{0, 1, 2\}$, $\{0, 2, 3\}$, $\{0, 3, 1\}$ (which correspond to the curl equations).

Turning to the Maxwell stress-energy-momentum tensor, we note that in 3 + 1 spacetime this can be written as,

$$\mathsf{T}_{\mu\nu} = \frac{1}{4\pi} \mathsf{F}_{\mu\lambda} \mathsf{F}_{\lambda\nu} + \frac{1}{16\pi} \eta_{\mu,\nu} \mathsf{F}_{\kappa\lambda} \mathsf{F}_{\kappa\lambda}, \tag{60}$$

where,

$$\mathsf{F}_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{61}$$

and $\eta_{\mu\nu}$ is the "metric" tensor with $\eta_{00} = 1$, $\eta_{ii} = -1$, and all other components 0.

However, 3 + 1 spacetime is a special case in this regard, as number of distinct field components in the electric and magnetic field tensors F_e or F_m, eq. (54), in $n + 1$ spacetime is n components with indices $0i$ and $n(n-1)/2$ components with indices ij for a total of $N_n = n(n+1)/2$ in $n+1$ electrodynamics; $N_1 = 1$, $N_2 = 3$, $N_3 = 6$, $N_4 = 10$, *etc.* The only case with the same number of components of the two types, *i.e.*, with $n = n(n-1)/2$, is $n = 3$; only in 3 + 1 spacetime do the components of F_e or F_m combine into a single 4-tensor (61). In any other spatial dimension than 3, the components of the electric and magnetic field tensors (54) are different physical entities, and the stress-energy-momentum tensor is,

$$\mathsf{T}_{\mu\nu} = \frac{1}{4\pi}\mathsf{F}_{e,\mu\lambda}\mathsf{F}_{e,\lambda\nu} + \frac{1}{16\pi}\eta_{\mu,\nu}\mathsf{F}_{e,\kappa\lambda}\mathsf{F}_{e,\kappa\lambda} + \frac{1}{4\pi}\mathsf{F}_{m,\mu\lambda}\mathsf{F}_{m,\lambda\nu} + \frac{1}{16\pi}\eta_{\mu,\nu}\mathsf{F}_{m,\kappa\lambda}\mathsf{F}_{m,\kappa\lambda}. \tag{62}$$

Finally, (in the present survey) the Lorentz-force (density) law in 3 + 1 spacetime is,

$$\mathbf{f} = \mathbf{f}_e + \mathbf{f}_m = \rho_e\mathbf{E} + \frac{\mathbf{J}_e}{c} \times \mathbf{B} + \rho_m\mathbf{B} - \frac{\mathbf{J}_m}{c} \times \mathbf{E}, \tag{63}$$

assuming that the electromagnetic fields in eq.(63) are the total fields $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$. To cast the Lorentz force into tensor notation, it is useful to introduce the dual tensors $\mathsf{F}^\star_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\mu\kappa\lambda}\mathsf{F}_{\mu\nu}$,

$$\mathsf{F}^\star_e = \begin{pmatrix} 0 & -B_{e,1} & -B_{e,2} & -B_{e,3} \\ B_{e,1} & 0 & E_{e,3} & -E_{e,2} \\ B_{e,2} & -E_{e,3} & 0 & E_{e,1} \\ B_{e,3} & E_{e,2} & -E_{e,1} & 0 \end{pmatrix}, \quad \mathsf{F}^\star_m = \begin{pmatrix} 0 & -E_{m,1} & -E_{m,2} & -E_{m,3} \\ E_{m,1} & 0 & -B_{m,3} & B_{m,2} \\ E_{m,2} & B_{m,3} & 0 & -B_{m,1} \\ E_{m,3} & -B_{m,2} & B_{m,1} & 0 \end{pmatrix}, \tag{64}$$

Then, we can define the total field tensors,

$$\mathsf{F} = \mathsf{F}_e + \mathsf{F}^\star_m, \qquad \mathsf{F}^\star = \mathsf{F}^\star_e + \mathsf{F}_m, \tag{65}$$

and the Lorentz-force law can be written as,

$$f_\mu = \mathsf{F}_{\mu\nu}\frac{J_{e,\nu}}{c} + \mathsf{F}^\star_{\mu\nu}\frac{J_{m,\nu}}{c} \qquad (3 + 1 \text{ spacetime}), \tag{66}$$

However, as seen above, only in 3 + 1 spacetime can the components of the field tensors $\mathsf{F}_{e,\mu\nu}$ and $\mathsf{F}_{m,\mu\nu}$ be combined into a single field tensor $\mathsf{F}_{\mu\nu}$ such that total 3-vector electric and magnetic fields can be defined as the sums of the 3-vector fields due to electric and magnetic charges, $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$. For other spatial dimensions we can only have,

$$f_\mu = \mathsf{F}_{e,\mu\nu}\frac{J_{e,\nu}}{c} + \mathsf{F}_{m,\mu\nu}\frac{J_{m,\nu}}{c} \qquad (n \neq 3). \tag{67}$$

which means that magnetic charges, if they exist, do not interact electromagnetically with electric charges (and so would be very hard to detect in apparatus made of electric charges). There is no distinction between the behavior of "electric" and "magnetic" charges when eq. (67) applies, so if both types of charges exist, it as if there were two types of electric charges with no interactions between the two types.

Note also that it is a logical possibility that the form (67) holds in 3 + 1 spacetime as well, which could be why magnetic charges have not been detected in apparatus based on the electromagnetic interaction of electric charges.[30]

[30]Magnetic charges that do not couple to electric charges, but have their own "magnetodynamics", are not candidates for the dark matter of galactic haloes, as the magnetodynamic interaction would permit clumping of magnetic-charge matter similar to the clumping of electric-charge matter via electrodynamics.

A.1 Electrodynamics in 1 Spatial Dimension

In 1 + 1 electrodynamics, with one spatial dimension, x, the field tensors (54) have components,

$$\mathsf{F}_e = \begin{pmatrix} 0 & -E_e \\ E_e & 0 \end{pmatrix}, \qquad \mathsf{F}_m = \begin{pmatrix} 0 & -B_m \\ B_m & 0 \end{pmatrix}. \tag{68}$$

There is no magnetic field associated with moving electric charge, and no electric field due to moving magnetic charge. That is, the electric field is only due to electric charge, and the magnetic field is only due to magnetic charge.

The relations (59) lead to only trivial equations, as there are now only two value for the indices $\{\lambda, \mu, \nu\}$ which must be all distinct to have a nontrivial relation. The relations (56) lead to the "Maxwell" equations for $S_1 = 2$,

$$\frac{\partial E_e}{\partial x} = 2\rho_e, \qquad \frac{\partial E_e}{\partial t} = -2J_e, \qquad \frac{\partial B_m}{\partial x} = 2\rho_m, \qquad \frac{\partial B_m}{\partial t} = -2J_m. \tag{69}$$

Only two of these four equations are independent, in view of charge conservation,

$$\frac{\partial \rho_e}{\partial t} = -\frac{\partial J_e}{\partial x}, \qquad \frac{\partial \rho_m}{\partial t} = -\frac{\partial J_m}{\partial x}. \tag{70}$$

Equations (69) indicate that the fields E_e and B_m are constant in both x and t in charge-free regions. Hence, there are no electromagnetic waves in charge-free regions in 1 + 1 electrodynamics, for which the constant c does not have the significance of the speed of electromagnetic waves.

If electric charge q is distributed uniformly between on the interval $[-dx/2, dx/2]$ then the charge density is $\rho = q/dx$, and the constant fields $\pm E_e$ outside the charge distribution are related by the first of eq. (69) by,

$$\frac{dE_e}{dx} = \frac{2E_e}{dx} = 2\rho_e = \frac{2q}{dx}, \tag{71}$$

so $E_e = q$, as previously argued in sec. 2.1.

The force density on electric and magnetic charge distributions is,

$$f = \rho_e E_e + \rho_m B_m. \tag{72}$$

There is no coupling between "electric" and "magnetic" charges.

A.2 Electrodynamics in 2 Spatial Dimensions

In 2 + 1 electrodynamics the field tensors (54) have components,

$$\mathsf{F}_e = \begin{pmatrix} 0 & -E_{e,1} & -E_{e,2} \\ E_{e,1} & 0 & -B_e \\ E_{e,2} & B_e & 0 \end{pmatrix}, \qquad \mathsf{F}_m = \begin{pmatrix} 0 & -B_{m,1} & -B_{m,2} \\ B_{m,1} & 0 & -E_m \\ B_{m,2} & E_m & 0 \end{pmatrix}. \tag{73}$$

The fields $\mathbf{E}_e$ and $\mathbf{B}_m$ are 2-vectors, while the fields B_e and E_m are scalars.

The force density on electric and magnetic charge distributions is,

$$\mathbf{f} = \rho_e \mathbf{E}_e + \frac{\mathbf{J}_{e,\perp} B_e}{c} + \rho_m \mathbf{B}_m + \frac{\mathbf{J}_{m,\perp} B_m}{c} \,. \tag{74}$$

As in any $n+1$ spacetime except $3+1$, there is no coupling between "electric" and "magnetic" charges, and no conceptual distinction between them.

B Appendix: Magnetic Field in Two Spatial Dimensions when the Current in a Loop Falls to Zero

All examples with axial symmetry about the z-axis in three spatial dimensions "descend"[31] to examples in two spatial dimensions in which the 3-d fields $\mathbf{E}_3 = E_\theta \,\hat{\boldsymbol{\theta}}$ and $\mathbf{B}_3 = B_z \,\hat{\mathbf{z}}$ correspond to the 2-d (vector) electric field $\mathbf{E}_2$ and the scalar magnetic field B_2.

As noted in sec. 2.2.1 above, this has a disconcerting implication for propagation of time-dependent effects in 2-d electrodynamics. Namely, a change in the source charges/currents at, say, $t = 0$ and $(r, \theta) = (r, 0)$ in a 2-d polar coordinate system leads to changes in the fields at (R, θ) for all times $t > R/c$ and not just at time $t = R/c$. This is because $r = 0$ in a 2-d example is equivalent to the entire z-axis in the 3-d, axially symmetric, equivalent example, where an observer at $r = R$ and $z = 0$ in a 3-d cylindrical coordinate system (r, θ, z) detects a change at $r = 0$, z and $t = 0$ at time $t = \sqrt{R^2 + z^2}/c$. Accordingly, the 2-d observer also detects a change in the fields at time $t = \sqrt{R^2 + z^2}/c$ for any value of z (although z is not a spatial coordinate for the 2-d observer, but only a parameter).

We illustrate this for the 2-d example of a circular current loop of radius a, where the current obeys,

$$I(t) = \begin{cases} I_0 & (t < 0), \\ 0 & (t > 0). \end{cases} \tag{75}$$

One might naïvely suppose that an observer at the origin would detect no magnetic field for $t > a/c$, but actually he would detect some magnetic field at arbitrarily large times.

To understand this, we consider the 3-d equivalent of this example, *i.e.*, an infinite solenoid of radius a with azimuthal current density I_0 per unit length in z. For $t < 0$, the 3-d magnetic field is purely axial,

$$B_z(t < 0) = \begin{cases} \frac{4\pi I_0}{c} & (r < a), \\ 0 & (r > a). \end{cases} \tag{76}$$

For $t > a/c$, an observer at the origin has not yet received the "message" that the current has dropped to zero for points with $|z| > \sqrt{c^2 t^2 - a^2}$, and supposes that the current still flows there. That is, the observer considers the magnetic field at time $t > 0$ to be $4\pi I_0/c$

[31] This terminology was introduced by Hadamard [32].

minus that due to a solenoid of radius a and axial extent $|z| < \sqrt{c^2t^2 - a^2}$. The latter field is,

$$\frac{4\pi I_0}{c}\frac{|z|}{\sqrt{a^2+z^2}} = \frac{4\pi I_0}{c}\sqrt{1-\frac{a^2}{c^2t^2}}\,, \tag{77}$$

recalling prob. 5.3 of [45]. The magnetic field at the origin for $t > a/c$ is therefore,

$$B_z(t > a/c) = \frac{4\pi I_0}{c}\left(1 - \sqrt{1-\frac{a^2}{c^2t^2}}\right) \approx \frac{4\pi I_0}{c}\frac{a^2}{2c^2t^2}\,, \tag{78}$$

where the approximation holds for $t \gg a/c$. For $t = 2a/c$, the magnetic field at the origin is still 0.13 of its steady value for $t < a/c$.

This behavior, deduced via a 3-d analysis, also applies to the 2-d version of the example, on changing the factor 4π to 2π.

C Appendix: Duality Transformations (Feb. 19, 2022)

If both electric and magnetic charges exist, Maxwell's equations in (3 + 1) dimensions can be written as,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho_e, \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_m, \quad -c\boldsymbol{\nabla}\times\mathbf{E} = \frac{\partial\mathbf{B}}{\partial t} + 4\pi\mathbf{J}_m, \quad c\boldsymbol{\nabla}\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + 4\pi\mathbf{J}_e. \tag{79}$$

These equations are invariant under the duality transformations,[32]

$$q_e \to q_m, \qquad q_m \to -q_e, \qquad \mathbf{E}\to\mathbf{B}, \qquad \mathbf{B}\to-\mathbf{E}, \tag{80}$$

in the sense that they take the same form but with electric charges replaced by magnetic charges, and *vice versa*.[33]

We can use the duality transformations (80) in (1 + 1) and (2 + 1) electrodynamics to obtain additional versions of electrodynamics there. However, as noted towards the end of Appendix A above, there is no coupling between electric and magnetic charges in (1 + 1) or (2 + 1) electrodynamics, so the additional versions of electrodynamics are effectively identical to their dual forms, being different only in notation.

C.1 (1 + 1) Electrodynamics

C.1.1 The Electrodynamics of Sec. 2.1.1

In sec. 2.1.1 above, we considered electrodynamics based on the "Maxwell" equations,

$$\frac{\partial E}{\partial x} = 2\rho_e, \qquad \frac{\partial E}{\partial t} = -2J_e. \tag{81}$$

[32]For a historical review of the concept of electromagnetic duality, see Appendix D of [44].

[33]In Gaussian units, electric and magnetic charges have the same dimensions, and electric and magnetic fields have the same dimensions.

According to the duality transformations (80), we can also consider (1 + 1) electrodynamics according to,

$$\frac{\partial B}{\partial x} = 2\rho_m, \qquad \frac{\partial B}{\partial t} = -2J_m. \tag{82}$$

This is essentially the same electrodynamics, but the term "electric" replaced by the term "magnetic".

C.1.2 The Electrodynamics of Footnote 7

In footnote 7 above, we considered electrodynamics based on the "Maxwell" equations,

$$\frac{\partial B}{\partial x} = \frac{1}{c}\frac{\partial E}{\partial t} + J_e, \qquad \frac{\partial E}{\partial x} = \frac{1}{c}\frac{\partial B}{\partial t}\,. \tag{83}$$

According to the duality transformations (80), we can also consider (1 + 1) electrodynamics according to,

$$-\frac{\partial E}{\partial x} = \frac{1}{c}\frac{\partial B}{\partial t} + J_m, \qquad \frac{\partial B}{\partial t} = -\frac{1}{c}\frac{\partial E}{\partial t}\,. \tag{84}$$

Again, the difference between the dual forms, here (83) and (84), is only a matter of notation.

C.2 (2 + 1) Electrodynamics

In eq. (23) of sec. 2.2 above, we considered (2 + 1) electrodynamics with the "Maxwell" equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 2\pi\rho_e, \qquad \boldsymbol{\nabla}_\perp\cdot\mathbf{E} = \frac{1}{c}\frac{\partial B}{\partial t}\,, \qquad \boldsymbol{\nabla}_\perp B = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{2\pi}{c}\mathbf{J}_e. \tag{85}$$

According to the duality transformations (80) we can also consider the electrodynamics according to,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 2\pi\rho_m, \qquad \boldsymbol{\nabla}_\perp\cdot\mathbf{B} = -\frac{1}{c}\frac{\partial E}{\partial t}\,, \qquad -\boldsymbol{\nabla}_\perp E = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \frac{2\pi}{c}\mathbf{J}_m. \tag{86}$$

Again, there is no physical difference between the dual forms, (85) and (86).

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