Field Energy of Two Spheres in Contact

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(July 25, 2022)

1 Problem

What is the electrostatic field energy of two spheres of radius R that are in contact, when each sphere carries a uniform surface charge density?

This problem was posed to the author by Christopher Provatidis.

2 Solution

The concept of electrostatic potential energy was first formulated by Poisson (1812) [1], as an analog of gravitational potential energy, noting that in both cases the basic force law has a $1/r^2$ behavior. The electric potential energy U of two charges q_1 and q_2 separated by distance d equals the work needed to bring these charges together from infinity (where their potential energy is zero),[1]

$$U = \frac{q_1 q_2}{d} = q_1 V_2 = q_2 V_1 \,, \tag{1}$$

in Gaussian units, where the electric potential V at distance d from charge q is $V = q/r$. For a continuous electric charge density ρ, the electric potential $V(\mathbf{r})$ at an observation point $\mathbf{r}$, and the total potential energy U, are given by,

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\mathrm{Vol}', \qquad U = \frac{1}{2}\int \rho V \, d\mathrm{Vol}. \tag{2}$$

Poisson later (1824) advocated an analogous magnetic potential energy [4], in a model that magnetism is due to magnetic charges/poles.[2]

The concept of magnetic-field energy was introduced by Maxwell (1856), pp. 63-64 of [6], based on Poisson's model of magnetic potential energy, together with the magnetic field equations $\boldsymbol{\nabla} \cdot \mathbf{B} = 4\pi\rho_m$ and $\mathbf{B} = -\boldsymbol{\nabla} V_m$. Then,

$$U_m = \frac{1}{2}\int \rho_m V_m \, d\mathrm{Vol} = \int \frac{B^2}{8\pi}\, d\mathrm{Vol}. \tag{3}$$

In Art. 631 of his *Treatise* (1873) [7], Maxwell gave the analogous relation for the electric field $\mathbf{E}$,

$$U_e = \frac{1}{2}\int \rho V \, d\mathrm{Vol} = \int \frac{E^2}{8\pi}\, d\mathrm{Vol}. \tag{4}$$

[1] Newton showed that the effect of gravity on an external particle by a spherical shell of mass is the same as if the mass were concetrated at its center. See Theorem XXXI, p. 239 of [2]. The concept of the gravitational potential was introduced by D. Bernoulli in 1747 [3]. For Bernoulli, kinetic energy (*vivarum*) was mv^2, so his graviational potential energy was $2m_1 m_2/d$.

[2] In 1820, Ampère [5] conjectured that all magnetism is due to electric currents rather than magnetic poles, and as far as we know today, Ampère was correct.

2.1 Two Point Charges Separated by Distance $2R$

Maxwell did not consider point charges, but if we do, the electric potential energy of charges q_1 and q_2 separated by distance $2R$ is $V = q_1q_2/2R$. However, the electric field energy of the system is, supposing the total electric field is the sum of that due to the two charges, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E_2}$,

$$U = \int \frac{E^2}{8\pi}\, d\text{Vol} = \int \frac{(E_1 + E_2)^2}{8\pi}\, d\text{Vol} = \int \frac{E_1^2}{8\pi}\, d\text{Vol} + \int \frac{E_2^2}{8\pi}\, d\text{Vol} + \int \frac{\mathbf{E}_1 \cdot \mathbf{E}_2}{4\pi}\, d\text{Vol} \qquad (5)$$

The last integral in eq. (5) is called the interaction energy, and equals $q_1q_2/2R$ for the case of two point charges. The integrals of E_i^2 represent the self energy of the point charges, and are infinite.

This indicates a difficulty with the concept of point charges in field theory, which it is convenient to ignore (as an early example of "renormalization").

Before turning to the more comfortable case of charges with a nonzero radius, we note a peculiarity if $q_1 = q = -q_2$, that the integral of the interaction energy is zero inside a sphere of radius R centered on the midpoint of the line of centers of the two point charges [8]. Then, the integral of the interaction energy outside this sphere equals the full interaction energy,

$$U_{\text{int}} = \int \frac{\mathbf{E}_1 \cdot \mathbf{E}_2}{4\pi}\, d\text{Vol} = \int_{>R} \frac{\mathbf{E}_1 \cdot \mathbf{E}_2}{4\pi}\, d\text{Vol} = -\frac{q^2}{2R}\,, \qquad (6)$$

for this special case.

2.2 Two Charged Spheres of Radius R That Touch

The electric potential of sphere i is q_i/r_i for $r_i > R$ (oustide sphere i), where $\mathbf{r}_i$ points from the center of sphere i to the observation point The electric fields $\mathbf{E}_i$ is zero inside sphere i, and equals $q_i\hat{\mathbf{r}}_i/r_i^2$ outside. The self energies are simply,[3]

$$U_i = \int_{r_i>R} \frac{\mathbf{E}_i^2}{8\pi}\, d\text{Vol} = \frac{q_i^2}{2R} = \frac{q_iV_i(r_i = R)}{2}\,. \qquad (7)$$

The interaction energy is difficult to calculate using the fields $\mathbf{E}_i$, but we can use the electric potentials of the two spheres to compute, via surface integrals over the two spheres,

$$U_{\text{int}} = \frac{1}{2}\left(\int_1 \sigma_1 V_2\, d\text{Area} + \int_2 \sigma_2 V_1\, d\text{Area}\right) = \int_1 \sigma_1 V_2\, d\text{Area} = \frac{q_1q_2}{4\pi R^2}\int_{-1}^{1} \frac{2\pi R^2\, d\cos\theta}{(5R^2 - 4R^2\cos\theta)^{1/2}}$$
$$= -\frac{q_1q_2}{4R}(5 - 4\cos\theta)^{1/2}\Big|_{-1}^{1} = \frac{q_1q_2}{2R}\,. \qquad (8)$$

where the uniform surface charge density is $\sigma_i = q_i/4\pi R^2$, and we have used Dwight 191.01 [10]. The interaction energy of the two spheres of radius R is the same as if the charge were concentrated at the centers of the spheres.[4]

---

[3]For the case of gravity, the self energies are negative, so the gravitational field cannot be a vector field (for which the self-field energy is positive). This was noted by Maxwell in sec. 82 of [9].

[4]This could be anticipated from our experience with the gravitational potential energy of spheres. Indeed, the results (7)-(9) hold even when the two spheres don't touch.

The total field energy is,

$$U = U_1 + U_2 + U_{\rm int} = \frac{q_1^2}{2R} + \frac{q_2^2}{2R} + \frac{q_1 q_2}{2R}\,. \tag{9}$$

For the special case that $q_1 = -q_2 = q$, the total field energy is $q^2/2R$, which is the negative of the interaction energy.

References

[1] S.D. Poisson, *Sur la Distribution de l'Électricité a la Surface des Corps Conducteurs*, Mém. Inst. Imp., 1 (1812), 163 (1813), kirkmcd.princeton.edu/examples/EM/poisson_12.pdf

[2] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686), kirkmcd.princeton.edu/examples/mechanics/newton_principia.pdf

[3] D. Bernoulli, *Commentationes de immutatione et extensione principii conservationisvirium vivarum, quae pro motu corporum coelestium requiritur*, Comm. Acad. Sci. Imp. Petr. **10**, 116 (1747), http://kirkmcd.princeton.edu/examples/mechanics/bernoulli_caisp_10_116_47.pdf

[4] S.D. Poisson, *Sur la Théorie du Magnétisme*, Mém. Acad. Roy. Sci. **5**, 247, 486 (1826), http://kirkmcd.princeton.edu/examples/EM/poisson_24.pdf

[5] A.M. Ampère, *Notes de M. Ampère sur les lectures qu'il a faites à l'Académie des Sciences*, J. Phys. **91**, 166 (1820), p. 166, http://kirkmcd.princeton.edu/examples/EM/ampere_jp_91_166_20.pdf

[6] J.C. Maxwell, *On Faraday's Lines of Force*, Trans. Camb. Phil. Soc. **10**, 27 (1858), kirkmcd.princeton.edu/examples/EM/maxwell_tcps_10_27_58.pdf

[7] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[8] A.C. Tort, *On the electrostatic energy of two point charges*, Rev. Bras. Ens. Fis. **36**, 3301 (2014), http://kirkmcd.princeton.edu/examples/EM/tort_rbef_36_3301_14.pdf

[9] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[10] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4$^{\rm th}$ ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf