

FINAL EXAM

PH 205

JAN. 14, 1983

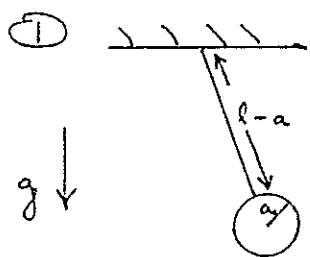
7:30 PM

TIME LIMIT : 3 HOURS

THE EXAM IS CLOSED BOOK, CLOSED NOTES, ETC.

DO ALL WORK YOU WISH GRADED IN THE EXAM BOOKLETS PROVIDED.

THE EXAM CONSISTS OF 6 PROBLEMS WORTH 10 POINTS EACH.



A PENDULUM IS IN THE FORM OF A SPHERICAL SHELL OF RADIUS  $a$  ATTACHED TO A ROD OF LENGTH  $l-a$ . YOU MAY IGNORE THE MASS OF THE ROD AND SHELL. THE SHELL IS

INITIALLY FILLED WITH ICE OF MASS  $M$ . LATER THE ICE MELTS,

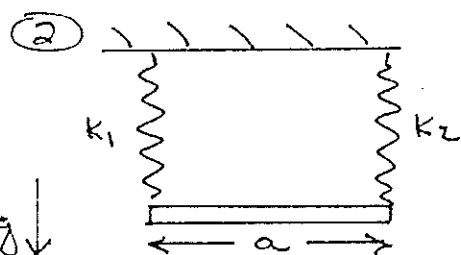
AND WE SUPPOSE THERE IS NO FRICTION BETWEEN THE WATER

AND THE SPHERICAL SHELL. CALCULATE THE RATIO OF THE

PERIOD OF THE ICE FILLED PENDULUM TO THAT OF THE WATER

FILLED PENDULUM. IGNORE ANY CHANGE IN DENSITY ON MELTING.

1 OPP IF  $\omega$  RATHER THAN PERIOD



TWO SPRINGS OF EQUAL REST LENGTH

SUPPORT A BAR OF MASS  $M$ , LENGTH  $a$

AS SHOWN.

a) SUPPOSE  $k_1 = k_2$  (= SPRING CONSTANTS). WHAT ARE THE

NORMAL FREQUENCIES OF OSCILLATION? CONSIDER ONLY

MOTION IN A VERTICAL PLANE, AND IN WHICH THE ENDS OF

THE BAR MOVE ESSENTIALLY VERTICALLY.

b) SUPPOSE  $k_1 \neq k_2$ . NOW WHAT ARE THE NORMAL FREQUENCIES?

WHAT ARE THE STRETCHES OF THE TWO SPRINGS AT

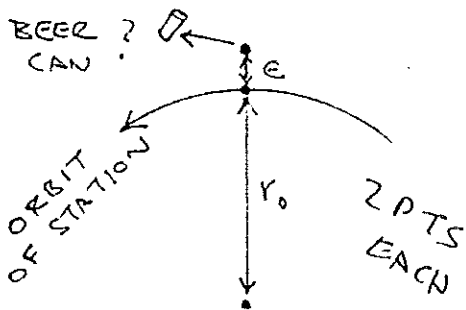
EQUILIBRIUM?

1 PT

5 PTS

1/

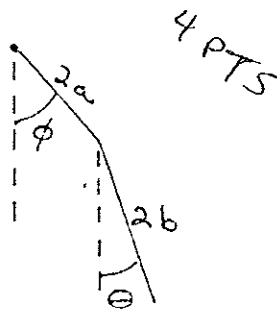
3 A SPACE STATION IS IN A CIRCULAR ORBIT ABOUT THE EARTH AT DISTANCE  $r_0$  FROM THE CENTER OF THE EARTH. AN ASTRONAUT ON A SPACE WALK IS AT DISTANCE  $r_0 + e$  ALONG THE LINE JOINING THE CENTER OF THE EARTH TO THE SPACE STATION. AFTER SOME PRACTICE THE ASTRONAUT FINDS THAT (S)HE CAN THROW A BEER CAN SO THAT IT MOVES IN THE PLANE OF THE ORBIT OF THE SPACE STATION, AND IT APPEARS TO ORBIT ABOUT THE SPACE STATION (ACCORDING TO AN OBSERVER ON THE STATION).



IN WHAT DIRECTION AND WITH WHAT SPEED SHOULD THE ASTRONAUT THROW THE BEER CAN? WHAT IS THE SIZE, SHAPE AND PERIOD OF THE ORBIT OF THE BEER CAN ABOUT THE SPACE STATION?

HINT: DO MOST OF YOUR ANALYSIS IN THE LAB FRAME.

4 SINGLE BAR EXERCISE A CHINESE TOY IS IN THE



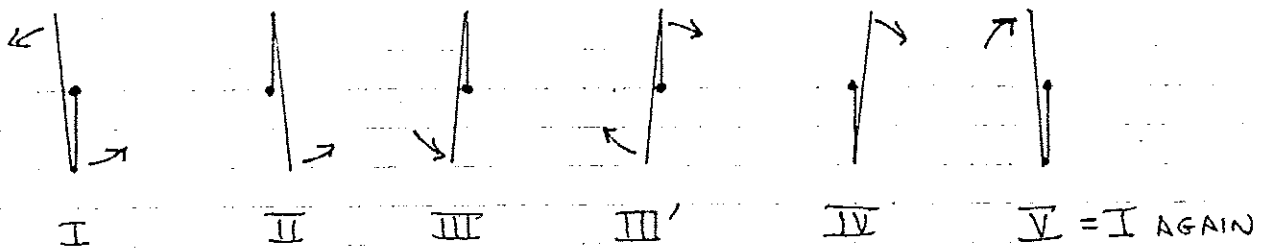
FORM OF A DOUBLE PENDULUM CONSISTING OF TWO RODS OF LENGTHS  $2a$  AND  $2b$  AND MASS  $m$  EACH, AS SHOWN. THE ANGLE  $\phi$  IS CONSTRAINED TO OBEY A KNOWN FUNCTION  $\phi(t)$ .

a) FIND THE EQUATION OF MOTION FOR ANGLE  $\theta$ .

I THINK YOU WILL AGREE THAT AN EXACT SOLUTION OF THIS EQUATION IS NOT ELEMENTARY.

4) b) MAKE AN APPROXIMATE, ELEMENTARY ANALYSIS OF ONE MODE OF OPERATION OF THE TOY:

6 PTS



A CYCLE STARTS IN CONFIGURATION I

FROM I TO II THE TWO BARS ROTATE TOGETHER WITH ANGULAR VELOCITY  $\omega$ .

FROM II TO III BAR a IS FIXED WHILE BAR b SWINGS FREE UNTIL CONFIGURATION III IS REACHED.

NOW THE CYCLE REVERSES:

FROM III' TO IV THE BARS ROTATE TOGETHER AT RATE  $\omega$ .

FROM IV TO V BAR a IS FIXED WHILE BAR b SWINGS FREE.

SUPPOSE  $\omega$  IS BARELY SUFFICIENT TO ALLOW STEP II  $\rightarrow$  III TO OCCUR. WHAT IS THE PERIOD OF THE WHOLE CYCLE I  $\rightarrow$  V? YOU MAY ASSUME  $a = b/2$ .

TRY OUT YOUR RESULT NUMERICALLY WITH  $b = 5\text{cm}$

FEEL FREE TO PLAY WITH THE TOY!

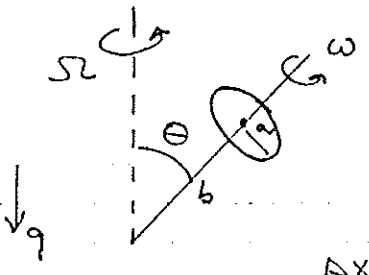
2 TO DETERMINE  $\omega$

2 TO DETERMINE  $\Omega$

1 TO ADD  $\omega$  UP

1 PT FOR NUMERICAL

⑤ A SYMMETRIC TOP IS IN THE FORM OF A HOOP OF



MASS  $M$ , RADIUS  $a$  WHOSE CENTER IS  
 A DISTANCE  $b$  FROM THE POINT WHERE  
 THE (MASSLESS) AXLE IS FIXED. THE  
 AXLE MAKES ANGLE  $\theta$  TO THE VERTICAL, AND  
 THE HOOP ROTATES AT ANGULAR VELOCITY  $\omega$  ABOUT THE  
 AXLE.

A POSSIBLE MOTION OF THE TOP IS STEADY PRECESSION ABOUT THE VERTICAL AT RATE  $\Omega$ .

ANALYZE THE SYSTEM IN A FRAME WHICH ROTATES ABOUT THE VERTICAL AT RATE  $\Omega$  TO FIND A RELATION BETWEEN  $\Omega$  AND  $\omega$ . WHAT IS THE MINIMUM  $\omega$  SUCH THAT STEADY PRECESSION CAN HOLD? ALSO INDICATE THE RELATION BETWEEN  $\Omega$  AND  $\omega$  IN THE LIMIT OF LARGE  $\omega$ .

TO COMPARE WITH THE RESULT IN THE NOTES, YOU MAY WISH TO GENERALISE YOUR RESULT TO THE CASE OF AN ARBITRARY SYMMETRIC TOP. (NOT FOR CREDIT)

2 FOR HI  $\omega$  LIMIT  
 1 FOR  $\omega$  MIN  
 $\Rightarrow$  7 TO SOLVE FOR  $\Omega$

6 BELOW IS A TRANSLATION OF THE BEGINNING OF A PAPER BY  
E. SCHRÖDINGER.

## Quantisation as a Problem of Proper Values (Part I)

(*Annalen der Physik* (4), vol. 79, 1926)

§ 1. IN this paper I wish to consider, first, the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary quantum conditions can be replaced by another postulate, in which the notion of "whole numbers", merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the *node-numbers* of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules.

The usual form of the latter is connected with the Hamilton-Jacobi differential equation,

$$(1) \quad H\left(q, \frac{\partial S}{\partial q}\right) = E.$$

A solution of this equation is sought such as can be represented as the sum of functions, each being a function of one only of the independent variables  $q$ .

Here we now put for  $S$  a new unknown  $\psi$  such that it will appear as a *product* of related functions of the single co-ordinates, i.e. we put

$$(2) \quad S = K \log \psi.$$

The constant  $K$  must be introduced from considerations of dimensions; it has those of *action*. Hence we get

$$(1') \quad H\left(q, \frac{K}{\psi} \frac{\partial \psi}{\partial q}\right) = E.$$

Now we do *not* look for a solution of equation (1'), but proceed as follows. If we neglect the relativistic variation of mass, equation (1') can always be transformed so as to become a quadratic form (of  $\psi$  and its first derivatives) equated to zero. (For the *one-electron* problem this holds even when mass-variation is not neglected.) We now seek a function  $\psi$ , such that for any arbitrary variation of it the integral of the said quadratic form, taken over the whole co-ordinate space,<sup>1</sup> is stationary,  $\psi$  being everywhere real, single-valued, finite, and continuously differentiable up to the second order. *The quantum conditions are replaced by this variation problem.*

First, we will take for  $H$  the Hamilton function for Keplerian motion, and show that  $\psi$  can be so chosen for *all positive*, but only for a *discrete set of negative values of  $E$* . That is, the above variation problem has a discrete and a continuous spectrum of proper values.

The discrete spectrum corresponds to the Balmer terms and the continuous to the energies of the hyperbolic orbits. For numerical agreement  $K$  must have the value  $h/2\pi$ .

The choice of co-ordinates in the formation of the variational equations being arbitrary, let us take rectangular Cartesian. Then (1') becomes in our case

$$(1'') \quad \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{2m}{K^2} \left(E + \frac{e^2}{r}\right) \psi^2 = 0;$$

$e$  = charge,  $m$  = mass of an electron,  $r^2 = x^2 + y^2 + z^2$ .

Our variation problem then reads

$$(3) \quad \delta J = \delta \iiint dx dy dz \left[ \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{2m}{K^2} \left(E + \frac{e^2}{r}\right) \psi^2 \right] = 0,$$

the integral being taken over all space. From this we find in the usual way . . . .

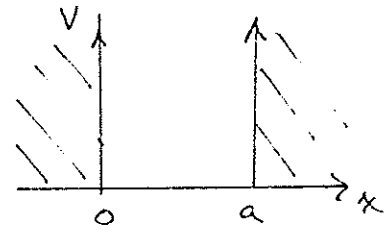
FOR WHAT ITS WORTH,  $S = \text{ACTION} = \int L dt$ , AND  
 $\frac{\partial S}{\partial q} = p = \text{GENERALISED MOMENTUM}$ . SEE L&L SEC 43.  
 THEN YOU CAN GO FROM SCHRÖDINGER'S (1') TO (1'')  
 BY FOLLOWING HIS SUGGESTION TO REPLACE  $p$  BY  $\frac{\hbar}{i} \frac{\partial \psi}{\partial x}$

WE CONSIDER A SLIGHT VARIATION ON  $\psi$  (1''):  
 ONE DIMENSIONAL MOTION, BUT SUBJECT TO AN ARBITRARY  
 FORCE DERIVED FROM POTENTIAL  $V(x)$ . THEN

5 PTS  $(1'') \rightarrow \left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{2M}{\hbar^2} (E + V(x)) \psi$

WITH THIS FORM OF THE INTEGRAND, CARRY OUT  
 THE VARIATION SUGGESTED IN (3) TO DERIVE  
 SCHRÖDINGER'S EQUATION.

CONSIDER THE 'INFINITE WELL'  
 POTENTIAL  $V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{ELSWHERE} \end{cases}$



4 PTS FOR E SOLVE SCHRÖDINGER'S EQUATION WITH THIS POTENTIAL,  
 SUPPOSING  $\psi$  IS CONTINUOUS AT  $x=0$  AND  $a$ .  
 CALCULATE THE ALLOWED VALUES OF THE ENERGY  $E$ .

THIS SOLUTION IS MEANT TO DESCRIBE THE  
 POSSIBLE MOTION OF A QUANTUM MECHANICAL PARTICLE  
 IN A BOX. THE POSITION OF THE PARTICLE IS UNCERTAIN  
 BY AN AMOUNT  $\sim a$ . THE MOMENTUM IS KNOWN - BUT  
 NOT ITS SIGN! IN THIS SENSE, THE UNCERTAINTY

1 FOR P

IN MOMENTUM IS  $\sim p$ . WHAT IS THE MINIMUM  
VALUE OF THE PRODUCT OF THE 'UNCERTAINTIES'  $pa$ ?

(NOTE:  $\psi = 0$  EVERYWHERE IS NOT CONSIDERED AN INTERESTING CASE.)