Ph 205 Final Exam Solutions

1989-90

1. Central force $\bar{F} = -\frac{\alpha}{r^2} e^{-\beta r} \hat{r}$

This can in principle be derived from a potential $V(r) = -\int_\infty^r \bar{F}\cdot d\bar{r} = \alpha \int_\infty^r \frac{e^{-\beta r}}{r^2} dr$

The key is that we do not actually have to do the integral!

The energy, in polar coordinates is $E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r) = \text{const}$

Angular momentum is also conserved: $L = m r^2 \dot{\theta} = \text{const}$

$$\therefore \quad E = \underbrace{\frac{1}{2} m \dot{r}^2}_{T_{eff}} + \underbrace{\frac{L^2}{2mr^2} + V(r)}_{V_{eff}}$$

The condition for an equilibrium circular orbit is $V'_{eff}(r_0) = 0$

The condition for stability of small oscillations is $V''_{eff}(r_0) \geq 0$

$$V'_{eff} = -\frac{L^2}{mr^3} + \frac{\alpha}{r^2} e^{-\beta r}$$

$$V''_{eff} = \frac{3L^2}{mr^4} - \frac{2\alpha}{r^3} e^{-\beta r} - \frac{\alpha\beta e^{-\beta r}}{r^2}$$

$$V'_{eff}(r_0) = 0 \Rightarrow \frac{L^2}{mr_0} = \alpha e^{-\beta r_0}$$

The equilibrium angular velocity Ω satisfies $L = m r_0^2 \Omega$

so $\boxed{m\Omega^2 r_0 = \frac{\alpha e^{-\beta r_0}}{r_0^2}}$ (which also follows from $F = ma$ at once)

$$V''_{eff}(r_0) = 3m\Omega^2 - m\Omega^2 r_0 \left(\frac{2}{r_0} + \beta\right) = m\Omega^2 (1 - r_0\beta)$$

$$\omega^2_{osc} = \frac{V''_{eff}(r_0)}{m} = \Omega^2 (1 - r_0\beta) \Rightarrow \boxed{\omega = \Omega\sqrt{1 - r_0\beta}}$$

Clearly for stability we must have $\boxed{r_0 < \frac{1}{\beta}}$

This contrasts with the gravitational force, for which stable orbits can have arbitrarily large r_0.

The nuclear force is 'short range.'

(2) STABILITY OF A 'DUMBELL' SATELLITE

A GOOD FIRST STEP IS TO DEDUCE THE EQUILIBRIUM ORIENTATION:

THIS IS UNSTABLE; IF ONE MASS ROTATES SLIGHTLY TOWARDS THE EARTH THE PULL OF GRAVITY ON IT INCREASES AND AMPLIFIES THE PERTURBATION

THIS IS STABLE: IF THE BAR ROTATES THE ATTRACTION ON THE ~~MASS~~ CLOSER TO EARTH PROVIDES A GREATER RESTORING TORQUE THAN THE DESTABILIZING TORQUE OF THE FAR MASS.

SO CONSIDER IS CASE LIKE

WE PRESENT TWO SOLUTIONS

METHOD A : CONSIDER THE SYSTEM IN A ROTATING FRAME WITH THE C.M. AT REST.

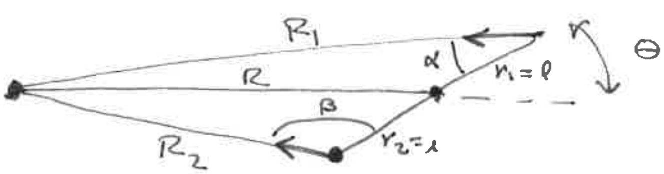
THEN WE MUST CONSIDER THE TORQUE EQUATION $I\ddot{\theta} = \sum$ TORQUE

HERE $I = 2m\ell^2$

NOW $\sum$ TORQUE $= \sum_{1,2} \bar{r}_i \times \bar{F}_i$

AND $\bar{F}_i = \bar{F}_{i,\text{GRAVITY}} \underbrace{- m\ddot{\bar{R}}}_{\text{C.M. MOTION}} \underbrace{- m\bar{\omega}_0 \times (\bar{\omega}_0 \times \bar{r}_i)}_{\text{CENTRIFUGAL FORCE, ALONG } \bar{r}_i} \underbrace{- 2m\bar{\omega}_0 \times \bar{v}_i}_{\text{CORIOLIS FORCE, ALONG } \bar{r}_i} + \underbrace{\bar{F}_{i,\text{ROD}}}_{\text{UNKNOWN CONSTRAINT FORCE!!}}$

NOTE THAT $\bar{r}_1 = -\bar{r}_2$, SO $\sum \bar{r}_i \times m\ddot{\bar{R}} = 0$. IF WE IGNORE $\sum \bar{r}_i \times \bar{F}_{i,\text{ROD}}$, ONLY GRAVITY PRODUCES A TORQUE IN THE ROTATING FRAME!



LAW OF COSINES:

$$R_{1,2} = R^2 + \ell^2 \pm 2R\ell\cos\theta$$

BY LAW OF SINES:

$$\frac{\sin\alpha}{R} = \frac{\sin\theta}{R_1}$$

$$\frac{\sin\beta}{R} = \frac{\sin\theta}{R_2}$$

$$\sum \text{TORQUE, GRAVITY} \approx r_1 \sin\alpha \cdot \frac{GMm}{R_1^2} - r_2\sin\beta \frac{GMm}{R_2^2}$$

$$= -GMm\,\ell\sin\theta\left(\frac{R}{R_2^3} - \frac{R}{R_1^3}\right)$$

$$\approx -GMm\,\ell\sin\theta\left[\frac{1-\frac{3}{2}(\ell^2/R^2 - 2\ell/R\cos\theta)+\dots}{R^2} - \frac{1-\frac{3}{2}(\ell^2/R^2 + 2\ell/R\cos\theta)+\dots}{R^2}\right]$$

$$\approx -\frac{6GMm\ell^2\sin\theta}{R^2}$$

NEAR EQUILIBRIUM, $\sin\Theta \approx \Theta$. ALSO, $F = ma$ FOR THE C.M. ORBIT TELLS US THAT $m\omega_0^2 R = \frac{GMm}{R^2}$

SO $\Sigma\tau \approx -6m\omega_0^2 l^2 \Theta = I\ddot{\Theta} = 2ml^2\ddot{\Theta}$

$$\ddot{\Theta} = -3\omega_0^2\Theta$$

$\Rightarrow$ FREQUENCY OF Θ OSCILLATIONS IS $\boxed{\omega = \sqrt{3}\,\omega_0}$

METHOD B USE LAGRANGE. HERE WE DON'T NEED TO KNOW $\vec{F}_{i,\text{ROD}}$!!

$L = T - V$ $\qquad T = T_{\text{OF C.M.}} + T_{\text{REL TO C.M.}}$

$$= \frac{1}{2} 2m\omega_0^2 R^2 + \frac{1}{2} 2ml^2 (\dot{\Theta} + \omega_0)^2$$

MINOR POINT: ANGULAR VELOCITY ABOUT THE C.M. IS $\dot{\Theta} + \omega_0$, NOT JUST $\dot{\Theta}$ IN AN INERTIAL FRAME

$$V = -GMm\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

AS BEFORE $\quad R_{1,2} = R\left(1 + \frac{l^2}{R^2} \pm \frac{2l}{R}\cos\Theta\right)^{1/2}$

$$\frac{1}{R_{1,2}} = \frac{1}{R}\left(1 - \frac{1}{2}\left(\frac{l^2}{R^2} \pm \frac{2l}{R}\cos\Theta\right) - \frac{1}{2}\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{l^2}{R^2} \pm \frac{2l}{R}\cos\Theta\right)^2 + \dots\right)$$

$$= \frac{1}{R}\left(1 \mp \frac{2l}{R}\cos\Theta + \frac{3}{8}\left(\frac{2l}{R}\cos\Theta\right)^2 + \dots\right)$$

SO $\frac{1}{R_1} + \frac{1}{R_2} \approx \frac{1}{R}\left(2 + 3\frac{l^2}{R^2}\cos^2\Theta + \dots\right)$

$$V \approx -\frac{3GMml^2\cos^2\Theta}{R^3} + \dots$$

$$= -3m\omega_0^2 l^2 \cos^2\Theta + \dots \qquad \text{USING} \quad m\omega_0^2 R = \frac{GMm}{R^2}$$

$$\approx 3m\omega_0^2 l^2 \Theta^2 + \dots \quad \text{IN SMALL ANGLE APPROX}$$

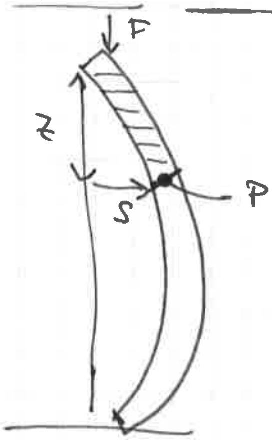
LAGRANGE $\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} = \frac{\partial L}{\partial \Theta} \Rightarrow 2ml^2\ddot{\Theta} = -6m\omega_0^2 l^2\Theta$

$$\ddot{\Theta} = -3\omega_0^2\Theta$$

AND THE FREQUENCY OF SMALL OSCILLATIONS IS $\boxed{\omega = \sqrt{3}\,\omega_0}$

WE INFER THAT THE $\vec{F}_{i,\text{ROD}}$ CAUSE NO RESTORING TORQUE!

(3) CHAPLIN'S CANE : ENERGY METHODS ARE CUMBERSOME!! INSTEAD,

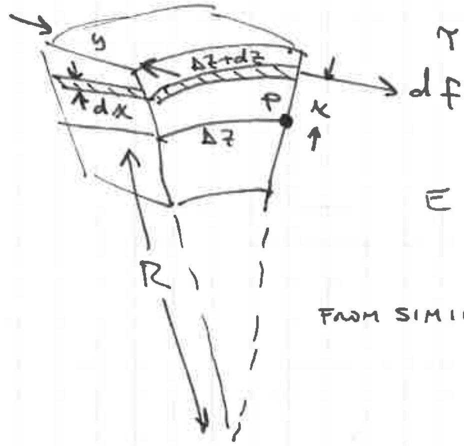


CONSIDER THE SHADED PIECE OF THE BEAM:

$$\sum \text{TORQUES ABOUT } P = 0 = Fs + \Upsilon \quad (1)$$

WHERE Υ IS THE TORQUE TRANSMITTED ACROSS A SLICE THRU THE BEAM AT P.

TO GET Υ, CONSIDER A WEDGE Δz LONG



$$\Upsilon = \int x\, df$$

df FROM YOUNG'S MODULUS:

$$E = \frac{df/A}{dz/\Delta z} \quad \text{WHERE } A = y\, dx$$

FROM SIMILAR TRIANGLES: $\dfrac{\Delta z}{R} = \dfrac{\Delta z + dz}{R + x}$

$$\approx \frac{\Delta z + dz}{R}\left(1 - \frac{x}{R}\right)$$

$$\approx \frac{\Delta z}{R} + \frac{dz}{R} - \frac{x \Delta z}{R^2}$$

SO $dz \approx \dfrac{x \Delta z}{R}$ AND $\Upsilon = \int x\, df = \int xy\, dx\, E \cdot \dfrac{x}{R} = \dfrac{E}{R} \underbrace{\int x^2 dx \cdot y}_{\equiv I}$

$$= \frac{EI}{R}$$

NOW R = RADIUS OF CURVATURE, SO $\dfrac{1}{R} \approx \dfrac{d^2 s}{dz^2}$

THEN (1) $\Rightarrow$ $Fs = -EI s''$ $\Rightarrow$ $s = A \sin \sqrt{\dfrac{F}{EI}}\, z$

BUT $\begin{cases} s(0) = 0 \\ s(L) = 0 \end{cases}$ SO MUST HAVE $\sqrt{\dfrac{F}{EI}} = \dfrac{n\pi}{L}$

FOR $\sqrt{\dfrac{F}{EI}} < \dfrac{\pi}{L}$ CANNOT SATISFY THE 2ND BOUNDARY CONDITION AND THE BEAM DOESN'T BEND!

CLEARLY $\boxed{F = EI\left(\dfrac{\pi}{L}\right)^2}$ IS THE CRITICAL CONDITION...

[THIS PROBLEM FIRST SOLVED BY L. EULER IN EARLY 1700'S.]

NOTE: $F \sim \dfrac{EI}{L^2}$ COULD BE DEDUCED FROM DIMENSIONAL ANALYSIS!

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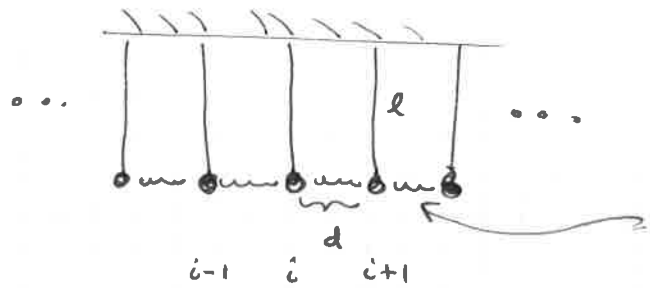
④ Slung Slinky

(Please remember to include gravity!)



Spring of constant A, rest length d

$s_i \equiv$ horizontal displacement of mass i

For mass i,

$$m\ddot{s}_i = \underbrace{-\frac{mg}{l}s_i}_{\text{pendulum force}} + \underbrace{A(s_{i+1}-s_i)}_{\text{force from spring to right}} - \underbrace{A(s_i - s_{i-1})}_{\text{force from spring to left}}$$

Take the continuum limit, introducing

$$\rho = \frac{m}{d}$$

$$s' = \frac{s_{i+1}-s_i}{d}$$

$$s'' = \frac{\frac{s_{i+1}-s_i}{d} - \frac{s_i - s_{i-1}}{d}}{d}$$

Then

$$\rho\ddot{s} = -\frac{\rho g}{l}s + \underbrace{Ad}_{\text{has dimensions of a force}}\, s''$$

[Recall $\rho\ddot{s} = Ts''$ for a string]

$$\ddot{s} = -\frac{g}{l}s + \frac{Ad}{\rho}s''$$

$$= -\omega_0^2 s + c^2 s'' \qquad \text{with } \omega_0^2 \equiv g/l \qquad c^2 = \frac{Ad}{\rho}$$

Try a wave solution: $s = \cos(kx - \omega t)$

Plug in to wave equation to get the dispersion equation:

$$\omega^2 = \omega_0^2 + c^2k^2$$

The lowest frequency is for long waves: $k = \frac{2\pi}{\lambda} \to 0$

$$\Rightarrow \quad \underline{\underline{\omega_{min} = \omega_0}} = \text{pendulum frequency!}$$

At ω_{min}, all masses swing together as free pendula, the springs do not stretch or compress!

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WHAT HAPPENS IF DRIVE THE SLINKY WITH $\omega < \omega_{MIN}$?

I.e. SUPPOSE $S(0,t) = S_0 \cos \omega t \qquad \omega < \omega_{MIN}$

TRY FOR A STANDING-WAVE SOLUTION AT FREQUENCY ω

$$S(x,t) = f(x) \cos \omega t$$

PLUG IN:
$$-\omega^2 f = -\omega_0^2 f + c^2 f''$$
$$f'' = \frac{\omega_0^2 - \omega^2}{c^2} f$$

IF $\omega_0 < \omega$ THEN $f \approx S_0 \cos \sqrt{\frac{\omega^2 - \omega_0^2}{c^2}}\, x$ AND WE HAVE LONG WAVES (THAT CAN BE CONSIDERED AS TRAVELLING WAVES AS WELL)

BUT IF $\omega_0 > \omega$, THEN $f = S_0 e^{-\sqrt{\frac{\omega_0^2 - \omega^2}{c^2}}\, x}$ AND THE DISTURBANCE AT THE ORIGIN DIES OUT IN SPACE.

THE FREQUENCY IS SO LOW THAT THE MOTION IS SLOWLY THAN FREE PENDULUM ACTION, AND THE SPRINGS CAN ADJUST TO A 'QUASI-STATIC' CONFIGURATION WITH ONLY A LOCALIZED REGION OF DISPLACEMENTS...