Ph 205 Solutions to Final Exam — Jan. 25, 1991

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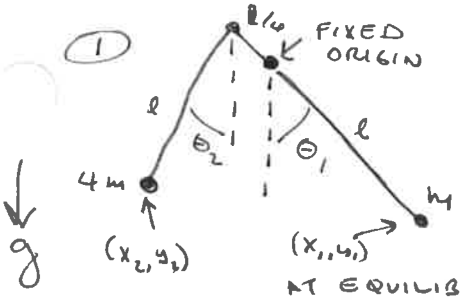

We should first note that if $\theta_2 = 0$, then the system is in equilibrium for ANY θ_1. This suggests that one normal mode will have zero frequency. Furthermore, we CANNOT assume $\theta_1 = 0$ at equilibrium for the other mode.

We now turn to Lagrange. Use θ_1 & θ_2 as 2 coords ($\Leftrightarrow$ 2 modes)

$$T_1 = \frac{m}{2} l^2 \dot\theta_1^2 \qquad V_1 = -mgl\cos\theta_1$$

$$x_2 = -\frac{l}{4}\sin\theta_1 - l\sin\theta_2 \qquad y_2 = +\frac{l}{4}\cos\theta_1 - l\cos\theta_2$$

$$\dot x_2 = -\frac{l}{4}\cos\theta_1\,\dot\theta_1 - l\cos\theta_2\,\dot\theta_2 \qquad \dot y_2 = -\frac{l}{4}\sin\theta_1\,\dot\theta_1 + l\sin\theta_2\,\dot\theta_2$$

$$T_2 = \frac{4ml^2}{2}\left[\frac{\dot\theta_1^2}{16} + \dot\theta_2^2 - \frac{1}{2}\cos(\theta_1+\theta_2)\,\dot\theta_1\dot\theta_2\right], \quad V_2 = 4mgl\left(\frac{\cos\theta_1}{4} - \cos\theta_2\right)$$

$$L = T - V = \frac{ml^2}{2}\left(\frac{5}{4}\dot\theta_1^2 + 4\dot\theta_2^2 - 2\cos(\theta_1+\theta_2)\dot\theta_1\dot\theta_2\right) + 4mgl\cos\theta_2$$

θ_1:
$$\frac{\partial L}{\partial\dot\theta_1} = ml^2\left(\frac{5}{4}\dot\theta_1 - \cos(\theta_1+\theta_2)\dot\theta_2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_1} = ml^2\left(\frac{5}{4}\ddot\theta_1 - \cos(\theta_1+\theta_2)\ddot\theta_2 + \sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)\dot\theta_2\right)$$

$$= \frac{\partial L}{\partial\theta_1} = ml^2\sin(\theta_1+\theta_2)\,\dot\theta_1\dot\theta_2$$

so $$\underline{\frac{5}{4}\ddot\theta_1 - \cos(\theta_1+\theta_2)\ddot\theta_2 + \sin(\theta_1+\theta_2)\dot\theta_2^2 = 0}$$

θ_2:
$$\frac{\partial L}{\partial\dot\theta_2} = ml^2\left(4\dot\theta_2 - \cos(\theta_1+\theta_2)(\dot\theta_1)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_2} = ml^2\left(4\ddot\theta_2 - \cos(\theta_1+\theta_2)\ddot\theta_1 + \sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2)\dot\theta_1\right)$$

$$= \frac{\partial L}{\partial\theta_2} = ml^2\sin(\theta_1+\theta_2)\,\dot\theta_1\dot\theta_2 - 4mgl\sin\theta_2$$

so $$\underline{-\cos(\theta_1+\theta_2)\ddot\theta_1 + 4\ddot\theta_2 + \sin(\theta_1+\theta_2)\dot\theta_1^2 = -4\omega_0^2\sin\theta_2}$$ where $\omega_0^2 \equiv \frac{g}{l}$

At equilibrium, $\dot\theta_1 = \dot\theta_2 = \ddot\theta_1 = \ddot\theta_2 = 0 \Rightarrow \sin\theta_2 = 0 \Rightarrow \theta_2 = 0, \pi$

As expected, there is no restriction on θ_1 for equilibrium!

"Clearly" only $\theta_2 = 0$ is stable.

Let
$$\left.\begin{aligned}\theta_1 &= \alpha + A\cos\omega t\\ \theta_2 &= B\cos\omega t\end{aligned}\right\}$$
be our trial solution, with A, B small

α is fixed, but arbitrary!

$\dot{\theta}_1^2 \sim A^2$ AND $\dot{\theta}_2^2 \sim B^2$, SO WE IGNORE THESE TERMS AS HIGHER ORDER.

$\cos(\theta_1+\theta_2) \approx \cos\alpha$ $\qquad \sin\theta_2 \approx B\cos\omega t$

AND WE CAN DIVIDE BY $\cos\omega t$ IN THE EQUATIONS OF MOTION TO GET

$$\theta_1: \quad -\frac{5}{4}\omega^2 A + \cos\alpha\, \omega^2 B = 0$$

$$\theta_2: \quad +\cos\alpha\, \omega^2 A - 4\omega^2 B = -4\omega_0^2 B$$

THE 1ST EQ. TELLS US THAT $\begin{cases} A = \frac{4}{5}\cos\alpha\, B & (\text{NOT } A=B!) \\ \underline{\text{OR}} \quad \underline{\omega = 0} & \underline{\text{THE TRIVIAL MODE}} \end{cases}$

THEN FOR $\omega \neq 0$, THE SECOND EQ BECOMES $\boxed{\omega = \dfrac{\omega_0}{\sqrt{1-\frac{1}{5}\cos^2\alpha}}}$

WHEN $\alpha = 0$ or π, $\omega = \sqrt{\frac{5}{4}\frac{g}{\ell}}$.

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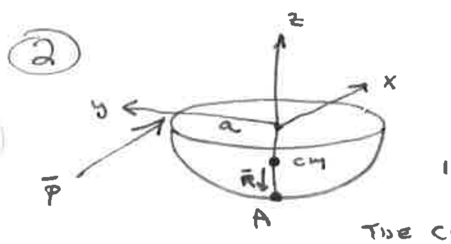


A 'THIN BOWL' IS A HALF SHELL NOT HALF OF A SOLID SPHERE.

IN GENERAL, $\bar{V}_A = \bar{V}_{cm} + \vec{\omega}\times\bar{R}$ WHERE $\bar{R}$ POINTS FROM THE CM TO A.

DUE TO THE IMPULSE $\bar{P} = P\hat{x}$, $\vec{V}_{cm} = \frac{\bar{P}}{M}$

TO FIND $\bar{\omega}$, WE NOTE THAT ANGULAR MOMENTUM ABOUT THE CM IS CONSERVED

$L_x = 0$, $\quad L_y = P(a-R) = I_y\,\omega_y \qquad L_z = -Pa = I_z\,\omega_z$

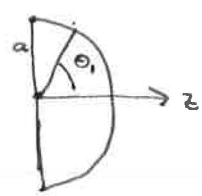
SO $\quad \bar{\omega} = \left(0, \frac{P(a-R)}{I_y}, -\frac{Pa}{I_z}\right)$

AND $\quad \vec{\omega}\times\bar{R} = \left(-P\frac{R(a-R)}{I_y}, 0, 0\right)$

$\bar{V}_A = \left(\frac{P}{M} - P\frac{R(a-R)}{I_y}, 0, 0\right)$

WE NEED R AND I_y TO FINISH

FIRST, WHERE IS THE C.M.?



$$M z_{cm} = \int_0^1 \underbrace{\rho\, 2\pi a^2\, d\cos\theta}_{dm} \cdot \underbrace{a\cos\theta}_{z}$$

$$= M\frac{a}{2} \qquad \text{NOTING } M = 2\pi a^2 \rho$$

SO $a - R = \frac{a}{2} \Rightarrow R = \frac{a}{2}$ ALSO.

NEXT, THE MOMENTS OF INERTIA.

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"CLEARLY" I_z (HALF SHELL OF MASS M) $= I_z$ (SHELL OF MASS M)

$$= M\left(\langle r_x^2\rangle + \langle r_y^2\rangle\right)$$

NOW FOR A SHELL, $\langle r_x^2\rangle + \langle r_y^2\rangle + \langle r_z^2\rangle = \langle r^2\rangle = r^2$
AND $\langle r_x^2\rangle = \langle r_y^2\rangle = \langle r_z^2\rangle$ $\Bigg\} \Rightarrow \langle r_x^2\rangle = \frac{r^2}{3}$

AND $I_z = \frac{2}{3} M a^2$ FOR SHELL OF RADIUS a.

ALSO, I_y (HALF SHELL, ABOUT DIAMETER THRU CENTER OF SHELL) $= I_z = \frac{2}{3} M a^2$

BY PARALLEL-AXIS THEOREM: $I_y = I_{y,cm} + M\left(\frac{a}{2}\right)^2$ $\boxed{\text{CAN'T USE PERPENDICULAR-AXIS THEOREM HERE!!}}$

SO $I_{y,cm} = Ma^2\left(\frac{2}{3} - \frac{1}{4}\right) = \underline{\frac{5}{12} M a^2}$ $\quad (= I_{x,cm})$

FINALLY, $V_{A,x} = \frac{P}{M} - \frac{P\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)}{\frac{5}{12}Ma^2} = \underline{\underline{\frac{2}{5}\frac{P}{M}}}$

$\left[\text{FOR A SOLID HEMISPHERE, } a - R = \frac{3}{8}a\text{, AND } I_{y,cm} = \frac{83}{320} Ma^2 \Rightarrow V_A = \frac{8}{83}\frac{P}{M}\right]$

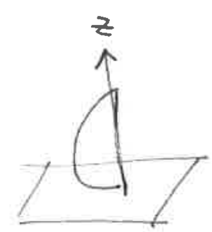
"SHOW THAT THE RIM OF THE BOWL NEVER TOUCHES THE HORIZONTAL SURFACE."

WE USE AN ENERGY ARGUMENT TO SEE THIS.

IN THE FRAME WHERE THE CM DOES NOT MOVE HORIZONTALLY, THE KINETIC ENERGY JUST AFTER THE IMPULSE IS $KE = \frac{1}{2}\bar{\omega}\cdot\bar{\bar{I}}\cdot\bar{\omega} = \frac{1}{2}\left(I_y\omega_y^2 + I_z\omega_z^2\right)$

FROM ABOVE, $I_x = I_y = \frac{5}{12}Ma^2$, $I_z = \frac{2}{3}Ma^2$
AND $\bar{\omega} = \left(0, \frac{Pa/2}{I_y}, \frac{Pa}{I_z}\right) = \left(0, \frac{6}{5}\frac{P}{Ma}, \frac{3}{2}\frac{P}{Ma}\right)$ $\Big\}$ JUST AFTER IMPULSE

SO $KE = \frac{P^2}{2M}\left(\frac{5}{12}\left(\frac{6}{5}\right)^2 + \frac{2}{3}\left(\frac{3}{2}\right)^2\right) = \frac{P^2}{2M}\left(\frac{3}{5} + \frac{3}{2}\right) = \frac{21}{20}\frac{P^2}{M}$



NOW SUPPOSE THE BOWL TIPPED UP TIL THE RIM TOUCHES

$KE > \frac{1}{2} I_z \omega_z^2$ CERTAINLY HOLDS.

IN THIS CONFIGURATION, $I_z = \frac{5}{12} Ma^2$ WHAT IS ω_z?

AS THE BOWL ROLLS AROUND, THE TORQUE ABOUT THE CM DUE TO THE NORMAL FORCE IS IN THE x-y PLANE $\Rightarrow L_z$ = CONSTANT.

FROM THE FIRST PART OF THE PROBLEM $|L_z| = Pa$, SO $\omega_z = \frac{L_z}{I_z} = \frac{12}{5}\frac{P}{Ma}$

AND $KE > \frac{1}{2}\frac{5}{12}Ma^2\left(\frac{12}{5}\frac{P}{Ma}\right)^2 = \frac{6}{5}\frac{P^2}{M} > KE_{\text{INITIAL}}$.

BUT, THE CM RISES AS THE BOWL TIPS, SO WE MUST HAVE $KE < KE_{\text{INITIAL}}$. $\Rightarrow$ RIM NEVER TOUCHES!

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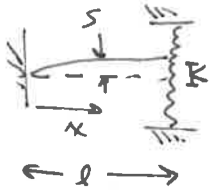


THE WAVE EQUATION OF THE STRETCHED STRING IS

$$\rho \ddot{s} = T s''$$

THE LEFT BOUNDARY CONDITION IS $s(0,t) = 0$

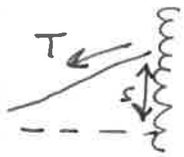
$\Rightarrow$ STANDING WAVES OF FORM $s = \sin kx \cos \omega t$

WHERE $\boxed{\omega = kc = k\sqrt{\frac{T}{\rho}}}$ (PLUG IN TO WAVE EQ. TO VERIFY!)

WAVE NUMBER, NOT SPRING CONSTANT K

AT THE RIGHT END WE HAVE A PECULIAR BOUNDARY CONDITION:

THE TUG OF THE STRING MUST COUNTERACT THE FORCE OF THE SPRING:



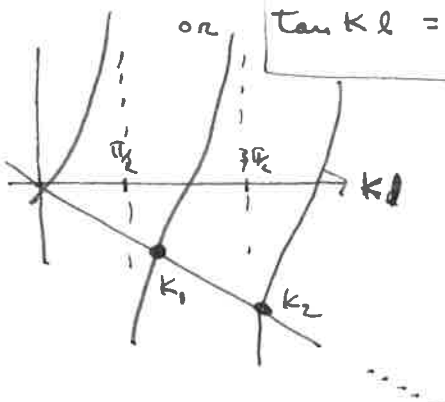
$$\Sigma F = 0 = \underbrace{-T s'}_{\text{STRING}} \underbrace{- K s}_{\text{SPRING}}$$

(WATCH SIGNS!)

(THE WORDING OF THE PROBLEM WAS AMBIGUOUS; SOME PEOPLE READ THAT THE STRING WAS TIED TO THE MIDDLE OF A SPRING OF CONSTANT K. THEN CONSIDER THIS AS 2 HALF SPRINGS EACH OF CONSTANT $2K$, THE TOTAL FORCE OF THE SPRING IS $-4Ks$)

PLUGGING IN $\quad -Tk \cos k\ell - K \sin k\ell = 0$

OR $\quad \boxed{\tan k\ell = -\frac{Tk}{K} = -\left(\frac{T}{K\ell}\right) k\ell}$



FOR $\frac{T}{K\ell} \gg 1$ WE SEE THAT $k_n = \frac{2n-1}{2}\frac{\pi}{\ell} - \epsilon$

$$\omega_n = \sqrt{\frac{T}{\rho}}\, k_n$$

$$\omega_2 \approx \frac{3\pi}{2}\sqrt{\frac{T}{\rho}}$$

SLIGHT DOWNTURN

MODE 1:

MODE 2:

SLIGHT UPTURN.

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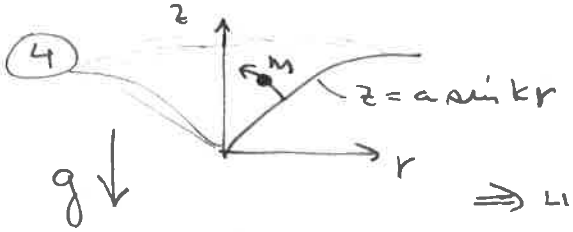

2 DEGREES OF FREEDOM : r, θ

BUT $L_z = mr^2\dot{\theta} = \text{CONST}$

$\Rightarrow$ LIKELY CAN REDUCE PROBLEM TO 1 DIMENSION.

E_{TOTAL} IS CONSERVED $= T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + mgz$

$$= \underbrace{\frac{1}{2}m\dot{r}^2\left(1 + \left(\frac{dz}{dr}\right)^2\right)}_{\frac{1}{2}m_{EFF}\dot{r}^2} + \underbrace{\frac{L^2}{2mr^2} + mga\sin kr}_{V_{EFF}(r)}$$

HOWEVER, $m_{EFF} = m\left(1 + \left(\frac{dz}{dr}\right)^2\right) = m\left(1 + a^2k^2\cos^2 kr\right) \neq$ CONSTANT

CAN WE USE OUR USUAL METHODS ?? FOR SMALL OSCILLATIONS ABOUT AN EQUILIBRIUM RADIUS r_0 $\quad m_{eff} \approx m\left(1 + a^2k^2\cos^2 kr_0\right)$ & THINGS SHOULD BE OK. (IF WORRIED, WRITE OUT LAGRANGE'S EQ. ---)

$$\frac{dV_{EFF}}{dr} = -\frac{L^2}{mr^3} + mgak\cos kr = 0 \text{ AT EQUILIBRIUM}$$

$$\Rightarrow \frac{L^2}{mr_0^3} = mgak\cos kr_0$$

WE ALSO WRITE $L = mr_0^2\Omega$ WHERE Ω = EQUILIBRIUM ANGULAR VELOCITY

SO $\Omega^2 = \frac{g}{r_0}ak\cos kr_0$ ETC.

$$K_{eff} = \left.\frac{d^2V_{EFF}}{dr^2}\right|_{r_0} = \frac{3L^2}{mr_0^4} - mgak^2\sin kr_0 = 3mga\frac{k}{r_0}\cos kr_0 - mgak^2\sin kr_0$$

FOR STABLE OSCILLATIONS, NEED $K_{eff} > 0 \Rightarrow \tan kr_0 < \frac{3}{kr_0}$ $\quad (kr_0 \lesssim 1.2)$

THE FREQUENCY OF OSCILLATIONS IS

$$\boxed{\omega^2 = \frac{K_{eff}}{m_{eff}} = \frac{3ga\frac{k}{r_0}\cos kr_0 - gak^2\sin kr_0}{1 + a^2k^2\cos^2 kr_0} = \frac{3\Omega^2\left(1 - \frac{kr_0}{3}\tan kr_0\right)}{1 + \left(\frac{r_0\Omega^2}{g}\right)^2}}$$

ETC