

MECHANICAL SIMILARITY ( $L \neq L$  SEC 10)

SOME FEATURES OF MOTION UNDER A CENTRAL FORCE CAN BE UNDERSTOOD WITHOUT SOLVING ANY DIFFERENTIAL EQUATIONS — BUT BY USING A KIND OF DIMENSIONAL ANALYSIS. THIS CAN ALSO GIVE US INSIGHT IN PROBLEMS INVOLVING MORE THAN 2 PARTICLES.

EXAMPLE OF SIMPLE DIMENSIONAL ANALYSIS. WHAT IS THE FUNCTIONAL

DEPENDENCE OF THE PERIOD OF A SIMPLE PENDULUM. IT MIGHT DEPEND ON  $g$ ,  $l$  = LENGTH OF PENDULUM,  $m$  = MASS OF BOB, AND  $\theta_{\max}$ .

BUT A SIMPLE PENDULUM HAS PERIOD INDEPENDENT OF  $m$  AND  $\theta_{\max}$

THUS  $T = f(g, l) \sim g^\alpha l^\beta$  BY HYPOTHESIS.

THE DIMENSIONS OF  $g$  ARE  $[g] = [l]/[t]^2$

SO WE REQUIRE  $[T] = [t] = [g]^\alpha [l]^\beta = \frac{[l]^{\alpha+\beta}}{[t]^{2\alpha}} \Rightarrow 2\alpha = -1 \quad \& \quad \beta = -\alpha$

HENCE  $T \sim \sqrt{l/g}$ . DIMENSIONAL ANALYSIS ALONE CANNOT FIX

THE NUMERICAL FACTOR OF  $2\pi$ , BUT THE ESSENTIALS ARE DETERMINED!

RETURNING TO OUR CENTRAL FORCE PROBLEM,

HOW DO PROPERTIES OF THE MOTION UNDER SUCH A POTENTIAL DEPEND ON THE SIZE OF THE ORBIT? MORE GENERALLY, CAN WE CHANGE THE SCALES OF LENGTH, TIME AND MASS IN SUCH A WAY THAT THE MOTION REMAINS THE 'SAME' IN SOME SENSE?

WE COULD ANSWER THIS BY LOOKING AT HOW THE EQUATIONS OF MOTION ARE ALTERED BY A SCALE CHANGE. SINCE THE EQUATIONS OF MOTION CAN BE GOTTEN FROM THE LAGRANGIAN,  $L = T - V$ , WE FIRST ASK, HOW IS  $L$  AFFECTED BY A SCALE CHANGE?

SUPPOSE ALL  $\vec{r}_i \rightarrow \alpha \vec{r}_i$   
 $t \rightarrow \beta t$   
 $m_i \rightarrow \gamma m_i$

THEN  $T = \frac{1}{2} \sum_i m_i \dot{r}_i^2 \rightarrow \gamma \left(\frac{\alpha}{\beta}\right)^2 T$  NOTING  $\dot{r} = \frac{dr}{dt} \rightarrow \frac{\alpha}{\beta} \frac{dr}{dt}$

IN ADDITION, SUPPOSE THE POTENTIAL  $V(\vec{r}_i, m_i)$  BEHAVES LIKE

$$V(\alpha \vec{r}_i, \gamma m_i) = \alpha^k \gamma^l V(\vec{r}_i, m_i)$$

[WE SAY THAT  $V$  IS 'HOMOGENEOUS' OF ORDER  $k$  IN  $\vec{r}$ , AND ORDER  $l$  IN  $m$ .]

$$\text{HENCE } L = T - V \rightarrow \gamma \left(\frac{\alpha}{\beta}\right)^2 T - \alpha^k \gamma^l V$$

$$\text{IF } \gamma \left(\frac{\alpha}{\beta}\right)^2 = \alpha^k \gamma^l \text{ THEN } L \rightarrow \alpha^k \gamma^l L$$

AND THE EQUATIONS OF MOTION WILL BE THE SAME! (SINCE THE EQUATIONS OF MOTION ARE UNAFFECTED IF  $L$  IS MULTIPLIED BY A CONSTANT.)

THIS MEANS THAT IF  $\bar{r} = f(t)$  IS A SOLUTION FOR MASS  $M$ , THEN  $\alpha \bar{r} = f(\beta t)$  IS A SOLUTION FOR MASS  $\gamma M$ .

THESE TWO SOLUTIONS ARE NOT IDENTICAL, BUT WE CAN SAY THEY ARE SIMILAR.

IF THE FORM OF THE LAGRANGIAN IS INVARIANT UNDER SCALE CHANGES OF MASS, LENGTH, AND TIME, THEN THE CHARACTER OF THE SOLUTIONS IS 'CONSERVED'. I.E., IF ELLIPSES ARE SOLUTIONS AT ONE SCALE, THEY WILL ALSO BE SOLUTIONS AT ANOTHER.

WE NOW RESTRICT OURSELVES TO THE CASE  $\gamma = 1$  (NO MASS SCALE CHANGE).

THEN WE NEED  $\frac{\alpha}{\beta} = \alpha^{k/2}$ , OR  $\beta = \alpha^{1-k/2}$  FOR SIMILARITY

TO HOLD. FOR EXAMPLE, IF  $\frac{t'}{t} = \beta = \alpha^{1-k/2} = \left(\frac{r'}{r}\right)^{1-k/2}$

THEN  $\frac{t^2}{r^{2-k}} = \text{CONSTANT}$  FOR SOLUTIONS OF SIMILAR SHAPE.

WE CAN IDENTIFY  $t$  AS PERIOD, AND  $r$  AS SOME CHARACTERISTIC LENGTH.

EXAMPLE: SPRING MOTION  $\Rightarrow k=2 \Rightarrow t^2 = \text{CONSTANT}$ .

I.E. THE PERIOD IS INDEPENDENT OF THE AMPLITUDE!

GRAVITY  $\Rightarrow k=-1 \Rightarrow \frac{t^2}{r^3} = \text{CONSTANT} \Leftrightarrow \text{KEPLER'S 3RD LAW}$ .

FOR A POTENTIAL,  $V \sim \frac{1}{r^k}$ ,  $\frac{T^2}{R^{2-k}}$  IS THE GENERALISED KEPLER LAW.

FOR GRAVITY WITH  $V = -\frac{GM_1 M_2}{r}$ ,  $V \rightarrow \alpha^{-1} \gamma^2 V$  IF THE MASS SCALE CHANGES ALSO. THEN  $\frac{\gamma \alpha^2}{\beta^2} = \gamma^2 / \alpha$  FOR SIMILARITY  $\Rightarrow \frac{\beta^2}{\alpha^3} = \frac{1}{\gamma}$   
OR  $T^2 / R^3 \sim 1 / \text{CHARACTERISTIC MASS}$ . (AS SEEN ON P.97)

VIRIAL THEOREM (L & L SEC 10)

THIS AMAZING RESULT RELATES THE AVERAGE KINETIC ENERGY TO THE AVERAGE POTENTIAL ENERGY. PERHAPS YOU ARE ALREADY FAMILIAR WITH THIS IDEA FOR THE CASES OF GRAVITY AND SPRING FORCES.

$$\langle T \rangle \equiv T_{\text{AVERAGE}} = \frac{1}{\tau} \int_0^{\tau} T(t) dt \quad \text{FOR TIME } \tau \rightarrow \infty$$

THE TRICK (DUE TO CLAUSIUS ~1870) IS TO WRITE

$$T = \frac{dG}{dt} + H$$

$$\text{SO } \langle T \rangle = \frac{1}{\tau} (G(\tau) - G(0)) + \langle H \rangle = \langle H \rangle \quad \text{IF } G \text{ IS WELL-BEHAVED.}$$

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} \sum_i \dot{\vec{p}}_i \cdot \dot{\vec{r}}_i = \frac{1}{2} \frac{d}{dt} \left( \sum_i \vec{p}_i \cdot \vec{r}_i \right) - \frac{1}{2} \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i \\ &= \frac{1}{2} \frac{d}{dt} \left( \sum_i \vec{p}_i \cdot \vec{r}_i \right) - \frac{1}{2} \sum_i \vec{F}_i \cdot \vec{r}_i \end{aligned}$$

$$\text{WE DEFINE } \mathcal{V} \equiv \text{VIRIAL} = \sum_i \vec{F}_i \cdot \vec{r}_i$$

$$\text{SO } \langle T \rangle = \frac{1}{2\tau} \left( \sum_i \vec{p}_i \cdot \vec{r}_i \Big|_{\tau} - \sum_i \vec{p}_i \cdot \vec{r}_i \Big|_0 \right) - \frac{1}{2} \langle \mathcal{V} \rangle$$

IF THE MOTION IS BOUNDED, THEN THE  $\vec{r}_i$  ARE FINITE AT ALL TIME, SO AS  $\tau \rightarrow \infty$

$$\langle T \rangle = -\frac{1}{2} \langle \mathcal{V} \rangle$$

VIRIAL THEOREM  
(CLAUSIUS, 1870)

$$\text{WE CAN GO FURTHER: } \vec{F}_i = -\frac{\partial V}{\partial \vec{r}_i}$$

ALSO, IF  $V$  IS A HOMOGENEOUS FUNCTION OF THE  $\vec{r}_i$  OF ORDER  $K$ , THEN

$$f(\alpha) \equiv V(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots, \alpha \vec{r}_n) = \alpha^K V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$

$$\text{SO } \frac{df}{d\alpha} = \sum_i \vec{r}_i \cdot \frac{\partial V(\alpha \vec{r}_i)}{\partial \vec{r}_i} = K \alpha^{K-1} V(\vec{r}_i)$$

$$\text{SETTING } \alpha = 1, \quad \sum_i \vec{r}_i \cdot \frac{\partial V}{\partial \vec{r}_i} = K V$$

PH 205 LECTURE 11

HENCE  $\mathcal{W} = \sum_i \vec{F}_i \cdot \vec{r}_i = - \sum_i \frac{\partial V}{\partial \vec{r}_i} \cdot \vec{r}_i = -K V$

SO  $\langle T \rangle = \frac{K}{2} \langle V \rangle$  (JACOBI, 1842)

FOR SPRINGS,  $K=2$ , AND  $\langle T \rangle = \langle V \rangle$ , WELL KNOWN FROM PH 103

FOR GRAVITY,  $K=-1$ , AND  $\langle T \rangle = -\frac{1}{2} \langle V \rangle$

$E = \langle T \rangle + \langle V \rangle = \frac{\langle V \rangle}{2} = -\langle T \rangle$

THIS IS TRUE FOR NON-CIRCULAR ORBITS AS WELL AS FOR CIRCULAR ORBITS: IF  $V = -\frac{\kappa}{r}$ ,  $\frac{m v^2}{r} = \frac{\kappa}{r^2}$  FOR A CIRCLE

SO  $T = \frac{1}{2} m v^2 = \frac{\kappa}{2r} = -\frac{1}{2} V$  !

FROM PH 103 YOU MAY RECALL THAT THIS RELATION ALSO HOLDS FOR  $N$  EQUAL MASSES AT THE CORNERS OF A ROTATING  $N$ -AGON.

THE SATELLITE PARADOX, PROB 6, SET 5, IS READILY EXPLAINED VIA THE VIRIAL THEOREM.

A PARADOX. WE SAY THAT MATTER IS BOUND TOGETHER BY ELECTRICAL ATTRACTION WITH THE COULOMB POTENTIAL,  $V \approx -\frac{\kappa}{r}$ . CONSIDER A CHUNK OF MATTER 'AT REST'. SINCE IT IS BOUND WE MUST HAVE  $V_{TOTAL} < 0$ . BY THE VIRIAL THEOREM,  $T = -V/2 > 0$ . HENCE THE CHUNK MUST HAVE SOME INTERNAL KINETIC ENERGY EVEN IF IT IS NOT ROTATING OR TRANSLATING. THUS CLASSICAL MECHANICS + COULOMB'S LAW PREDICTS THAT MATTER IS NOT SIMPLE INSIDE, BUT MUST HAVE SOME STRUCTURE CAPABLE OF INTERNAL MOTION.

SUPPOSE THE CHUNK IS SET IN MOTION.  $T$  INCREASES. BY THE VIRIAL THEOREM,  $E$  AND  $V$  MUST DECREASE. THIS IS THE PARADOX!

AS THE CHUNK OF MATTER MOVES IT MAY DEFORM. SINCE  $V \sim -\frac{\kappa}{r}$ , IF SOME (OR ALL) OF THE 'INTER-ATOMIC' DISTANCES DECREASE, THEN  $V$  DECREASES, AND THE VIRIAL THEOREM COULD BE SATISFIED.

THIS IS A SORT OF PRECURSOR TO THE THEORY OF RELATIVITY - WHEN AN OBJECT MOVES IT MUST SHRINK SOMEWHAT!

EXAMPLE THE VIRIAL THEOREM WAS INVENTED AS A TOOL IN THE KINETIC THEORY OF GASES. WE DON'T KNOW THE POTENTIAL OF THE FORCE BETWEEN MOLECULES, BUT WE CAN WRITE

$$\langle T \rangle = -\frac{1}{2} \langle \mathcal{V} \rangle = -\frac{1}{2} \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle$$

IN A MONOATOMIC GAS  $\langle T \rangle = U = \text{INTERNAL ENERGY} = \frac{3}{2} n K T$

$n = \#$  OF MOLECULES

$K = \text{BOLTZMANN'S CONSTANT}$

$T = \text{TEMPERATURE.}$

FOR A GAS IN A BOX,  $\mathcal{V} = \mathcal{V}_{\text{INTERNAL}} + \sum_i \vec{F}_{\text{WALL}} \cdot \vec{r}_i$

$\vec{F}_{\text{WALL}}$  IS  $\perp$  TO THE WALL, OF COURSE.

CONSIDER JUST  $\sum_{\substack{\text{WALLS} \\ 1 \in Z}} \vec{F}_x \cdot \vec{r} = \sum_{\text{WALL 1}} F_x (x_1 - x_2)$

NOW  $F_x = -P \text{ dAREA}$

WHERE  $P = \text{PRESSURE}$

ALSO  $x_1 - x_2 = l$

SO  $\sum_x F_x (x_1 - x_2) = -P \int dA = -P \int d\text{vol} = -PV$  ( $V = \text{VOLUME}$ )

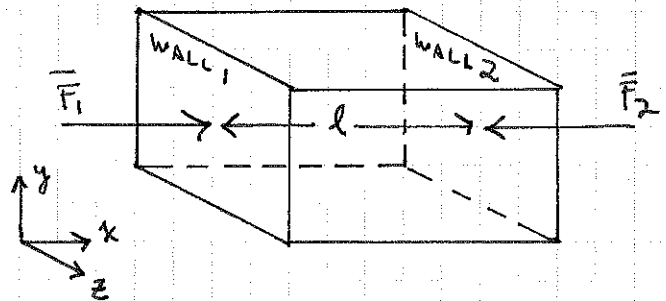
ADD THE  $y$  AND  $z$  PIECES,  $\mathcal{V} = \mathcal{V}_{\text{INTERNAL}} - 3PV$

THUS  $\frac{3}{2} n K T = -\frac{1}{2} (\mathcal{V}_{\text{INTERNAL}} - 3PV)$

OR  $PV = n K T + \frac{1}{3} \mathcal{V}_{\text{INTERNAL}}$

IN AN IDEAL GAS,  $\mathcal{V}_{\text{INTERNAL}}$  VANISHES!

FOR A NON-IDEAL GAS, AN APPROXIMATE GAS LAW CAN BE CONSTRUCTED BY MAKING GUESSES AS TO  $\mathcal{V}_{\text{INTERNAL}}$  ○○○○



COLLISIONS (L&L CHAP 4, B&O SECS 3-3, 4, 5, 6 ; 4-7) 118

WE TURN TO A RELATIVELY STRAIGHT FORWARD BRANCH OF CLASSICAL MECHANICS WHICH FINDS PROMINENCE IN MODERN ATOMIC AND SUB-ATOMIC PHYSICS: THE STUDY OF COLLISIONS.

WHEN DEALING WITH 'ELEMENTARY PARTICLES' WHOSE LIVES ARE VERY SHORT, THE ONLY KINDS OF EXPERIMENTS POSSIBLE ARE THOSE IN WHICH WE WATCH THE PARTICLES DISINTEGRATE, OR COLLIDE WITH ONE ANOTHER. HENCE THIS TINY CORNER OF CLASSICAL MECHANICS EXPANDS INTO A MAJOR TOPIC OF CONTEMPORARY PHYSICS.

WE SHALL CONSIDER 2 TOPICS:

- 1) HOW DO CONSERVATION LAWS RESTRICT THE POSSIBLE OUTCOME OF A DISINTEGRATION OR COLLISION?
- 2) WHAT IS THE STATISTICAL DISTRIBUTION OF THE RESULTS OF A DISINTEGRATION OR COLLISION, SUPPOSING THE FORCE DURING THE INTERACTION IS KNOWN?

WE SHALL CONSIDER THE CASE OF DISINTEGRATIONS FIRST, AS BEING SIMPLER THAN COLLISIONS, BUT ILLUSTRATING THE ESSENTIAL CONCEPTS. PART OF THE SIMPLICITY IS THAT CLASSICAL MECHANICS CANNOT REALLY EXPLAIN WHY A PARTICLE SHOULD DISINTEGRATE AT ALL!

OUR DISCUSSION WILL BE KINEMATICAL - I.E. CONCERNING PROPERTIES OF THE MOTION WHICH ARE INDEPENDENT OF THE DYNAMICS OF THE FORCES INVOLVED.

DISINTEGRATIONS (L&L SEC 16)

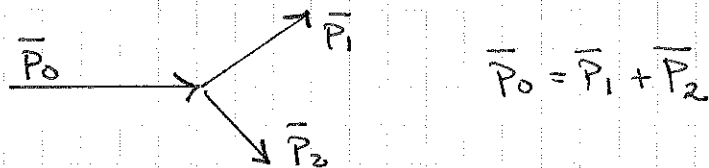
A PARTICLE OF MASS  $M_0$  SPONTANEOUSLY DISINTEGRATES INTO TWO PARTICLES OF MASSES  $M_1$  AND  $M_2$ .

EX: NUCLEAR FISSION  $U_{235} \rightarrow Th_{231} + He_4$

WE NOW CONSIDER THE 3 CONSERVATION LAWS.

a) MOMENTUM

SINCE THERE ARE NO EXTERNAL FORCES, MOMENTUM IS CERTAINLY CONSERVED.



A USEFUL TRICK IS TO CONSIDER THE PROBLEM IN THE C.M. FRAME.

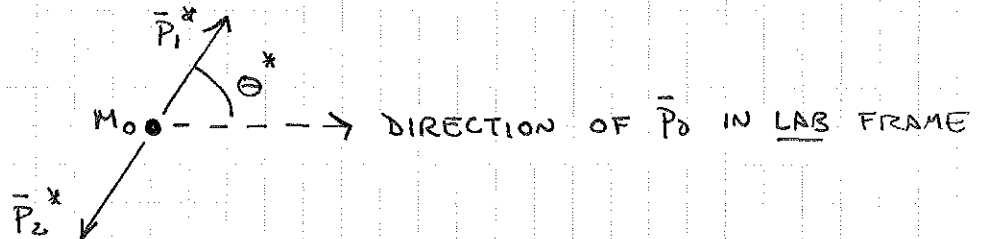
IN THIS FRAME  $m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0 \Rightarrow \vec{P}_1 + \vec{P}_2 = 0 \Rightarrow \vec{P}_0 = 0$

I.E. C.M. FRAME  $\Leftrightarrow$  REST FRAME OF THE INITIAL PARTICLE.

[ WE WILL LABEL QUANTITIES MEASURED IN THE C.M. FRAME WITH A \* ]

SO IN THE C.M. FRAME  $\vec{P}_1^* + \vec{P}_2^* = 0$  OR  $\vec{P}_1^* = -\vec{P}_2^*$

OUR PICTURE IS



THIS IS INDEPENDENT OF THE VALUES OF  $m_1$  AND  $m_2$ . OF COURSE, THE MOMENTA  $|\vec{P}_1^*| \neq |\vec{P}_2^*|$  ARE INDEPENDENT OF THE ANGLE  $\theta^*$  BETWEEN  $\vec{P}_1^*$  AND THE LABORATORY DIRECTION OF  $\vec{P}_0$ .

b) ENERGY. IF A PARTICLE CAN DISINTEGRATE WE MAY ANTICIPATE THAT IT HAS SOME INTERNAL STRUCTURE, AND HENCE COULD HAVE SOME INTERNAL ENERGY.

WE ALSO SUPPOSE THAT WE LOOK AT THE FINAL PARTICLES ONLY AFTER THEY ARE SO FAR APART THAT ANY POTENTIAL ENERGY BETWEEN THEM NO LONGER VARIES - IN EFFECT  $V_{12} = 0$ . (IF  $V_{12}(r)$  STILL VARIES, THERE IS STILL A FORCE BETWEEN 1 AND 2, AND WE SHOULD SAY THAT THE DISINTEGRATION IS NOT YET OVER).

NOTING THAT  $p = mv =$  MAGNITUDE OF MOMENTUM, THE ENERGY OF A PARTICLE IS THEN

$$E_{TOT} = E_{INT} + \frac{1}{2} m v^2 = E_{INT} + \frac{p^2}{2m}$$

CONSERVATION OF ENERGY THEN TELLS US

$$E_0 + \frac{p_0^2}{2m_0} = E_1 + \frac{p_1^2}{2m_1} + E_2 + \frac{p_2^2}{2m_2}$$

# PH 205 LECTURE 11

IN THE C.M. FRAME THIS BECOMES

$$E_0 = E_1 + \frac{P_1^{*2}}{2m_1} + E_2 + \frac{P_2^{*2}}{2m_2}$$

WHERE WE SUPPOSE  $E_{INTERNAL}$  IS INDEPENDENT OF THE FRAME.  
(NOT TRUE IN SPECIAL RELATIVITY!)

BUT IN THE C.M. FRAME  $P_1^* = P_2^* \equiv P^*$ , SO

$$Q \equiv E_0 - E_1 - E_2 = \frac{P^{*2}}{2} \frac{m_1 + m_2}{m_1 m_2} = \frac{P^{*2}}{2\mu}$$

THUS  $P^{*2} = 2\mu Q$  WITH  $\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{REDUCED MASS}$

AND  $Q = \text{INTERNAL ENERGY LIBERATED IN THE DISINTEGRATION.}$

OF COURSE, IF  $Q < 0$ , THE DISINTEGRATION CAN'T TAKE PLACE.  
IF  $Q$  IS KNOWN, IT IS A CONVENIENT SINGLE PARAMETER TO DESCRIBE THE DISINTEGRATION.

## c) ANGULAR MOMENTUM

IF THE INTERNAL FORCES DURING DISINTEGRATION ARE REASONABLY WELL BEHAVED, WE EXPECT ANGULAR MOMENTUM CONSERVATION. BUT IF THE PARTICLES HAVE INTERNAL ENERGY, WE MIGHT SUPPOSE THEY HAVE INTERNAL ANGULAR MOMENTUM. THEN CONSERVATION OF ANGULAR MOMENTUM WOULD PRIMARILY TELL US A RELATION AMONG THE INTERNAL ANGULAR MOMENTA. A CLASSICAL VIEW OF POINT PARTICLES CANNOT ACCOMMODATE ANY INTERNAL ANGULAR MOMENTUM, SO WE IGNORE THIS POSSIBILITY FOR NOW.

THEN, HOWEVER, ANGULAR MOMENTUM CONSIDERATIONS ADD NOTHING TO OUR STUDY OF DISINTEGRATIONS.

## d) TRANSFORMATION FROM C.M. FRAME BACK TO THE LAB.

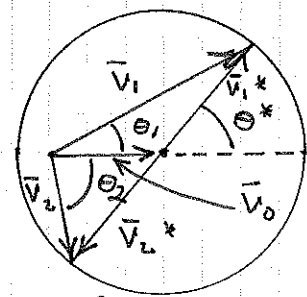
IN THE C.M. FRAME THE DECAY IS COMPLETELY DESCRIBED BY  $P^* = 2\mu Q$ , AND THE DIRECTION OF  $\vec{P}_1^*$ .

WE CANNOT LEARN ABOUT THIS DIRECTION FROM CLASSICAL MECHANICS. BUT IF IT IS GIVEN, WE CAN SHOW HOW TO TRANSFORM IT TO THE LAB FRAME.



IF  $\vec{p}_0 = m_0 \vec{v}_0$ , THEN  $\vec{v}_0 =$  VELOCITY OF THE C.M. AS SEEN IN THE LAB.

THEN  $\vec{v}_1 = \vec{v}_1^* + \vec{v}_0$  WITH  $\vec{p}_1^* = m_1 \vec{v}_1^*$   
 $\vec{v}_2 = \vec{v}_2^* + \vec{v}_0$   $\vec{p}_2^* = m_2 \vec{v}_2^*$

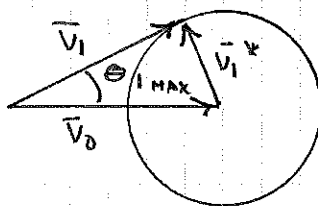


$m_1 = m_2$  →

AND  $v_1 \sin \theta_1 = v_1^* \sin \theta^*$   
 $v_1 \cos \theta_1 = v_0 + v_1^* \cos \theta^*$

$\tan \theta_1 = \frac{\sin \theta^*}{1 + \frac{v_1^* \cos \theta^*}{v_0}}$  ETC.

IF  $v_0 > v_1^*$ , THEN  $\theta_1$  HAS A MAXIMUM LESS THAN  $180^\circ$ .



$\sin \theta_{1 \text{ MAX}} = \frac{v_1^*}{v_0}$

ETC, ETC.

## STATISTICAL DISTRIBUTIONS

WE ARE UNABLE TO COMPLETELY SPECIFY THE RESULT OF A SINGLE DISINTEGRATION BY A CALCULATION FROM KNOWN INITIAL QUANTITIES. - THE ANGLE OF THE DECAY IN THE C.M. FRAME REMAINS ARBITRARY.

NATURE FORCES US TO ADOPT A STATISTICAL DESCRIPTION - WE CAN ONLY GIVE A STATEMENT OF HOW OFTEN CERTAIN ANGLES WILL OCCUR IF MANY DISINTEGRATIONS ARE OBSERVED.

WE INTRODUCE SOME OF THE LANGUAGE AND NOTATION COMMONLY USED TO DESCRIBE PHENOMENA STATISTICALLY.

SUPPOSE  $x$  REPRESENTS SOME MEASURABLE QUANTITY. THEN IF WE SAY THAT  $f(x)$  IS THE PROBABILITY OF OBSERVING  $x$ , WE MEAN MORE PRECISELY THAT IF  $N$  OBSERVATIONS ARE MADE, THE NUMBER IN WHICH VARIABLE  $x$  IS FOUND TO LIE BETWEEN  $x$  AND  $x+dx$  IS  $dN = N f(x) dx$ .

THIS SUPPOSES  $f$  IS NORMALISED TO 1 :  $\int f(x) dx = 1$ .

STRICTLY SPEAKING, THE PROBABILITY OF OBSERVING ANY GIVEN VALUE OF  $x$  IS ZERO - IF  $x$  IS A CONTINUOUS VARIABLE AND ONLY A COUNTABLE NUMBER OF EXPERIMENTS ARE MADE.

IT IS IMPORTANT TO KEEP IN MIND THE MEANING OF  $dN = N f(x) dx$  WHEN DISCUSSING A PROBABILITY DISTRIBUTION.  $f(x)$  ITSELF IS A SHORTHAND NOTATION, ONLY  $f(x) dx$  FOR SOME INTERVAL  $dx$  HAS A PHYSICAL MEANING.

WITH THESE SUBTLETIES IN MIND, WE OFTEN USE THE NOTATION  $\frac{dN}{dx} = N f(x)$  OR  $f(x) = \frac{1}{N} \frac{dN}{dx}$  ETC.

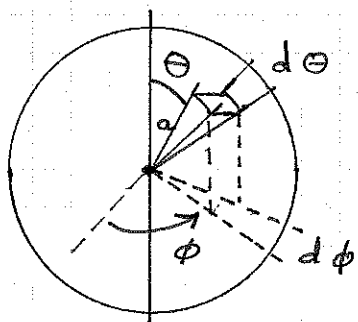
THIS DOES NOT MEAN  $N$  IS A FUNCTION OF  $x$  IN THE USUAL SENSE. IT IS JUST ANOTHER WAY OF ABBREVIATING OUR DEFINITION OF PROBABILITY GIVEN ABOVE.

### ISOTROPIC DECAY

WE CONSIDER A SIMPLE POSSIBILITY - ALL DECAY ANGLES ARE EQUALLY LIKELY IN THE C.M. FRAME.

HOW CAN WE DESCRIBE THIS IN OUR NEW NOTATION?

WE CHOOSE A SPHERICAL COORDINATE SYSTEM.



IF WE COUNT THE NUMBER OF DISINTEGRATIONS IN WHICH PARTICLE 1 PASSES THRU A GIVEN AREA  $dA$  ON A SPHERE OF RADIUS  $a$ , THEN WE EXPECT

$$dN = N \frac{dA}{4\pi a^2}$$

$$\text{OR } f(\text{AREA}) = \frac{1}{N} \frac{dN}{dA} = \frac{1}{4\pi a^2}$$

IN SPHERICAL COORDS,  $dA = (a d\theta)(a \sin\theta d\phi) = a^2 \sin\theta d\theta d\phi$

$$\text{SO } dN = N \frac{\sin\theta}{4\pi} d\theta d\phi$$

THUS WE COULD ALSO WRITE  $f(\theta, \phi) = \frac{1}{N} \frac{dN}{d\theta d\phi} = \frac{\sin\theta}{4\pi}$

OR NOTING THAT  $|\sin \theta d\theta| = |d\omega \theta|$  WE COULD WRITE

$$dN = N \frac{d\omega \theta d\phi}{4\pi} \Rightarrow f(\omega \theta, \phi) = \frac{1}{N} \frac{dN}{d\omega \theta d\phi} = \frac{1}{4\pi}$$

WE SEE THAT OUR EXPRESSION FOR  $f$  DEPENDS ON OUR CHOICE OF THE DIFFERENTIAL INTERVAL OVER WHICH OUR OBSERVATION OF  $dN$  IS TO BE MADE.

A COMMON DEFINITION IS

$$d\Omega = \sin \theta d\theta d\phi = d\omega \theta d\phi = \text{ELEMENT OF SOLID ANGLE}$$

$$\text{THEN } f(\Omega) = \frac{1}{N} \frac{dN}{d\Omega} = \frac{1}{4\pi}$$

ANOTHER COMMON PRACTICE IS TO INTEGRATE OVER  $\phi$ , IF ALL ANGLES  $\phi$  ARE EQUALLY LIKELY.

$$f(\omega \theta) = \int f(\omega \theta, \phi) d\phi = \int \frac{d\phi}{4\pi} = \frac{1}{2}$$

$$\text{HENCE } f(\omega \theta) = \frac{1}{N} \frac{dN}{d\omega \theta} \Rightarrow dN = \frac{1}{2} N d\omega \theta$$

IN SUMMARY, WE HAVE LEARNED THAT ISOTROPIC IN ANGLE  
 $\iff$  UNIFORM DISTRIBUTION OVER  $\omega \theta$  AND  $\phi$  (BUT NOT  
 UNIFORM IN  $\theta$ ).

IN PARTICULAR,  $f(\omega \theta) = \frac{1}{2}$  SINCE  $\omega \theta$  RUNS FROM  $-1$  TO  $+1$ .

CHANGE OF VARIABLE IF WE WISH THE DISTRIBUTION IN ANY OTHER VARIABLE, WE INVOKE

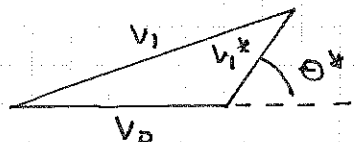
THE CHAIN RULE:

$$f(x) = \frac{1}{N} \frac{dN}{dx} = \frac{1}{N} \frac{dN}{d\omega \theta} \frac{d\omega \theta}{dx} = f(\omega \theta) \frac{d\omega \theta}{dx} \text{ ETC.}$$

$$\text{REMEMBER, } \frac{d\omega \theta}{dx} = \frac{1}{\frac{dx}{d\omega \theta}} \dots$$

EXAMPLE  $K = T_1 = \frac{1}{2} m_1 v_1^2 = \underline{\text{LAB KINETIC ENERGY OF PARTICLE 1}}$ .  
 [CONTINUATION OF P. 116 FE]

WE AGAIN USE THE LABEL  $\Theta^*$  TO REMIND US THAT IT IS ONLY IN THE C.M. FRAME THAT THE ANGULAR DISTRIBUTION IS ISOTROPIC.



$$v_1^2 = v_0^2 + v_1^{*2} + 2v_0 v_1^* \cos \Theta^*$$

$$\frac{dv_1^2}{d\cos \Theta^*} = 2v_0 v_1^*$$

$$\frac{dT_1}{d\cos \Theta^*} = m_1 v_0 v_1^*$$

$$\text{so } \frac{dN}{dT_1} = \frac{dN}{d\cos \Theta^*} \cdot \frac{1}{m_1 v_0 v_1^*} = \frac{N}{2} \cdot \frac{1}{m_1 v_0 v_1^*} \Rightarrow f(T_1) = \frac{1}{2m_1 v_0 v_1^*} = \text{CONST.}$$

(CAN YOU VERIFY THAT  $T_{\text{MAX}} - T_{\text{MIN}} = 2m_1 v_0 v_1^*$ , SO THAT  $f$  IS PROPERLY NORMALISED?)

EXAMPLE: AN OTHER TYPICAL QUESTION CONCERNS THE LABORATORY ANGULAR DISTRIBUTION:

$$\frac{dN}{d\Omega} = \frac{dN}{d\cos \Theta d\phi} = \frac{dN}{d\cos \Theta^* d\phi^*} \frac{d\cos \Theta^*}{d\cos \Theta} \frac{d\phi^*}{d\phi} = \frac{1}{4\pi} \frac{d\cos \Theta^*}{d\cos \Theta}$$

SINCE  $\phi^* = \phi$ .

HOWEVER, CALCULATION OF  $\frac{d\cos \Theta^*}{d\cos \Theta}$  IS USUALLY LENGTHY,

BUT STRAIGHT FORWARD. SEE PROB 2, P 44 L & L.