Ph 205 Lecture 17

Motion of a Rigid Body

[References for this and the next 2 lectures:
B & O Secs 5-5, 6, 7, 6-5, 6 ⋯ 13 ; L & L secs 31-38]

We recall several features of rigid-body motion discussed in Lecture 2.

- A rigid body has 6 degrees of freedom of motion: 3 of translation and 3 of rotation. We now wish to deal with the full complexity of rigid-body rotations.

- The 6 equations of motion are:

$$\frac{d\bar{P}}{dt} = \bar{F}_e \quad \text{and} \quad \frac{d\bar{L}}{dt} = \bar{N}_e \qquad (e = \text{external})$$

- The most general motion of a rigid body is a translation plus a rotation (Chasles' theorem). If we divide by dt to convert displacements into velocities, Chasles' theorem says that the velocity of any point in the body can be written $\bar{v} = \bar{v}_{cm} + \bar{\omega} \times \bar{r}$

  where $\bar{r}$ is measured with the C.M. as origin.

- The total angular momentum of a rigid body is

$$\bar{L} = \sum_i \bar{r}_i \times m_i \bar{v}_i = \sum_i m_i \bar{r}_i \times (\bar{v}_{cm} + \bar{\omega} \times \bar{r}_i) = M\bar{R}_{cm} \times \bar{V}_{cm} + \sum_i m_i \left(r_i^2 \bar{\omega} - (\bar{r}_i \cdot \bar{\omega}) \bar{r}_i \right)$$

  which is not simple in general.

- A major simplification occurs if the axis of rotation is a symmetry axis. Then we can write $\bar{L} = I\bar{\omega}$

  with $I = \sum_i m_i \rho_i^2$, ρ_i = distance of particle i to the axis.

Before dealing with the general case, we remind ourselves of the marvels of even the 'simple' case.

Gyroscopic Motion

A 'gyroscope' is any rigid body rotating about a symmetry axis with large angular velocity.

By 'large', we mean large enough that any changes in $\bar{L}$ due to external torques ($d\bar{L}/dt = \bar{N}_e$) are small compared to $\bar{L}$, so $\bar{L} = I\bar{\omega}$ still holds to a good approximation.

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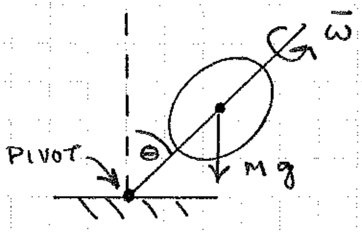
EXAMPLE STEADY PRECESSION OF A SYMMETRIC TOP



$$\bar{L} = I\bar{\omega}$$

WE CALCULATE THE TORQUE ABOUT THE PIVOT POINT (NOT THE C.M.) $\bar{N}_0 = \bar{R} \times m\bar{g} \Rightarrow N_0 = Rmg \sin\theta$

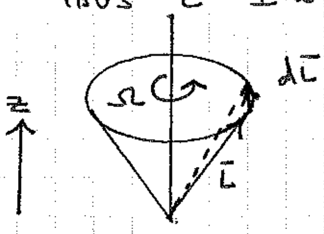
R = DISTANCE FROM PIVOT TO C.M.

WE ASSUME STEADY MOTION IN WHICH THE C.M. DOES NOT MOVE VERTICALLY ($\Rightarrow$ VERTICAL FORCE ON PIVOT $= mg$)

IF SO, θ = CONSTANT $\Rightarrow N_0$ = CONSTANT IN MAGNITUDE.

$$\frac{d\bar{L}}{dt} = \bar{N}_0 = \bar{R} \times m\bar{g} \Rightarrow \frac{d\bar{L}}{dt} \perp \text{ TO } \bar{R} \text{ AND HENCE } \perp \text{ TO } \bar{L} \Rightarrow |\bar{L}| = \text{CONSTANT}$$

THUS $\bar{L} = I\bar{\omega}$ MOVES IN A CONE OF ANGLE θ, AND WE CAN WRITE $\frac{d\bar{L}}{dt} = \bar{\Omega} \times \bar{L}$



NOW $\bar{L} = I\bar{\omega} = \frac{I\omega}{R}\bar{R}$ OR $\bar{R} = \frac{R}{I\omega}\bar{L}$

SO $\frac{d\bar{L}}{dt} = \bar{R} \times m\bar{g} = \frac{mgR}{I\omega}\hat{z} \times \bar{L} \equiv \bar{\Omega} \times \bar{L}$ $\qquad [\bar{g} = -g\hat{z}]$

WHERE $\bar{\Omega} = \frac{mgR}{I\omega}\hat{z}$ = PRECESSION ANGULAR VELOCITY.

CERTAIN APPROXIMATIONS HAVE BEEN MADE IN DERIVING THIS RESULT:

- SINCE THE TOP IS PRECESSING ABOUT THE VERTICAL AXIS, WE MIGHT EXPECT IT IS MORE CORRECT TO WRITE

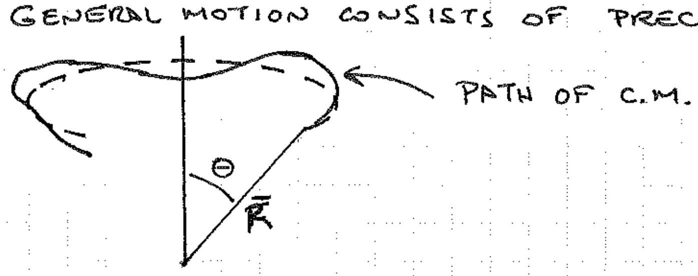
  $\bar{L} = I\bar{\omega} + K\bar{\Omega}$ $\qquad$ K SOME CONSTANT.

  WE HAVE IGNORED THE 2ND TERM IF ω IS 'LARGE', SINCE $\Omega \sim \frac{1}{\omega}$.

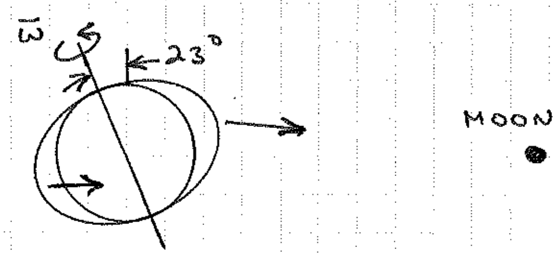
- THE VERTICAL FORCE ON THE PIVOT MIGHT NOT BE EXACTLY mg.

  $\Rightarrow$ VERTICAL MOTION OF THE C.M. $\Leftrightarrow$ VARIATION OF θ

  INDEED FOR FINITE $\bar{\omega}$ WE WILL FIND LATER THAT THE GENERAL MOTION CONSISTS OF PRECESSION ($\bar{\Omega}$) + OSCILLATION IN θ (NUTATION)



EXAMPLE: PRECESSION OF THE EQUINOXES



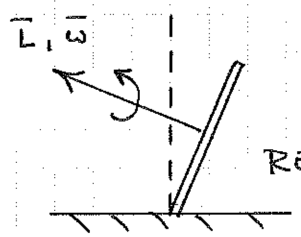
WE HAVE SEEN (SET 8) THAT THE EARTH'S CENTRIFUGAL BULGE IS MUCH LARGER THAN THE TIDAL BULGE. THE EARTH'S AXIS MAKES AN ANGLE OF ABOUT 23° TO THE 'VERTICAL' TO THE PLANE OF THE MOON'S ORBIT. THE TIDAL FORCE OF THE MOON (AND SUN) EXERTS A TORQUE ON THE CENTRIFUGAL BULGE, CAUSING THE EARTH'S AXIS TO PRECESS ABOUT THE 'VERTICAL' WITH A PERIOD OF ABOUT 26000 YEARS. WHAT IS THE SENSE OF THE PRECESSION COMPARED TO THAT OF THE ROTATION $\bar{\omega}$?

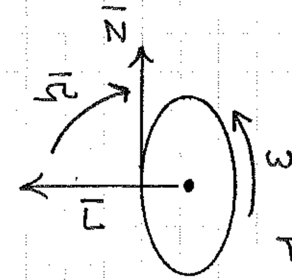
EXAMPLE: STABILITY OF A BICYCLE

SOME, BUT NOT ALL, OF THE STABILITY OF A BICYCLE IS DUE TO THE GYROSCOPIC ACTION OF THE SPINNING WHEELS.

IF THE BICYCLE FALLS SIDEWAYS TO THE RIGHT, YOU JUST STEER TO THE RIGHT $\Rightarrow$ MOTION IN A CIRCLE $\Rightarrow$ CENTRIFUGAL FORCE TO THE LEFT $\Rightarrow$ RESTORING FORCE TO BRING THE BICYCLE BACK TO THE VERTICAL! GYROSCOPE ACTION PLAYS NO ROLE IN THIS. IF THE BICYCLE IS SLOWLY MOVING, YOU WILL HAVE TO MAKE VIOLENT TURNS TO THE LEFT & RIGHT TO MAINTAIN STABILITY!

NOW SUPPOSE YOU RIDE THE BICYCLE "NO HANDS" $\Rightarrow$ NO STABILITY DUE TO STEERING.





NOW IF THE BIKE FALLS TO THE RIGHT, A TORQUE ABOUT THE C.M. DEVELOPS (DUE TO THE NORMAL FORCE OF THE ROAD) $\Rightarrow$ PRECESSION OF THE SPINNING WHEEL TO THE RIGHT $\Rightarrow$ MOTION OF C.M. IN A CIRCLE $\Rightarrow$ DESIRED CENTRIFUGAL FORCE IS GENERATED TO RESTORE THE BIKE TO THE VERTICAL.

THIS WORKS BETTER WHEN YOU RIDE FAST – HIGHER SPEEDS ARE NEEDED TO RIDE 'NO HANDS' THAN WITH HANDS ON.

IN PRACTICE YOU CAN ALSO SHIFT YOUR C.M. FROM SIDE TO SIDE WHICH HELPS STABILIZE THE BIKE IN A NON-GYROSCOPIC MANNER.

ON A MOTORCYCLE AT HIGH SPEEDS THE GYROSCOPE EFFECT BECOMES MORE IMPORTANT – YOU MUST LEAN INTO A TURN TO BE ABLE TO TURN MORE SHARPLY – GAINING THE EXTRA 'TURNING POWER' VIA GYROSCOPIC PRECESSION.

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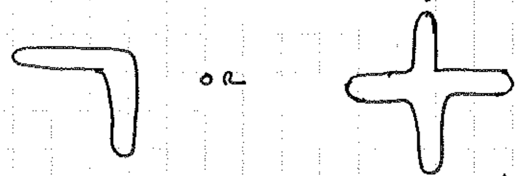
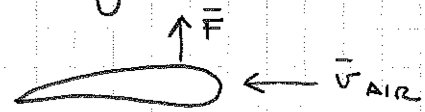
EXAMPLE: THE BOOMERANG (B & O SEC 5.6)

THE RETURNING FLIGHT OF THE BOOMERANG IS A GYROSCOPIC EFFECT!

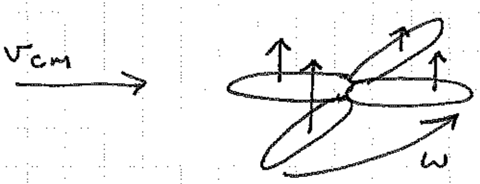
A BOOMERANG CONSISTS OF 2, 3, OR 4 BLADES, EACH SHAPED LIKE AN AIRPLANE WING

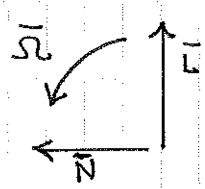


EACH BLADE HAS A CROSS-SECTION LIKE → WHICH CAUSES A LIFT FORCE ⊥ TO $\bar{v}$ AS IT FLIES THRU THE AIR.



IF THE BOOMERANG IS SPINNING AS IT FLIES, THE LIFT IS GREATER ON THE BLADE MOVING FORWARD THAN ON THE BLADE MOVING BACKWARD ⟹ NET TORQUE





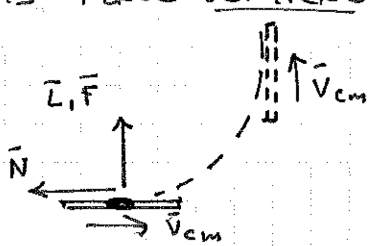
THE LIFT FORCE CAUSES A DEFLECTION OF THE C.M. MOTION WHICH IS ⊥ TO THE PLANE OF THE BOOMERANG.

THE LIFT TORQUE CAUSES THE PLANE OF THE BOOMERANG TO PRECESS ABOUT AN AXIS IN THAT PLANE, AND ⊥ TO $\bar{v}_{cm}$

TO GET A CIRCULAR (⇒ RETURNING) PATH OF THE C.M., THE DEFLECTION AND PRECESSION MUST MATCH, SO THAT $\bar{v}_{cm}$ ALWAYS REMAINS IN THE PLANE OF THE BOOMERANG.

THIS IS EASIEST TO ARRANGE IF YOU THROW THE BOOMERANG WITH ITS PLANE VERTICAL

TOP VIEW:



$\bar{F}$ DEFLECTS

$\bar{N}$ CAUSES PRECESSION

B & O CALCULATE THAT FOR A 4-BLADED BOOMERANG YOU MUST GIVE IT SPIN $\omega \sim \sqrt{\frac{3}{2}}\,\frac{v}{R}$ (R = BLADE LENGTH)

i.e., A SLIGHT TOPSPIN.

THE INERTIA TENSOR

WE NOW CONSIDER THE GENERAL EXPRESSION FOR THE ANGULAR MOMENTUM OF A RIGID BODY ABOUT ITS C.M.

$$\bar{L} = \sum_i m_i \bar{r}_i \times \bar{v}_i = \sum_i m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i) = \sum_i m_i \left(r_i^2 \bar{\omega} - (\bar{r}_i \cdot \bar{\omega}) \bar{r}_i \right)$$

FOR A CONTINUOUS MASS DISTRIBUTION THIS BECOMES

$$\bar{L} = \int \rho \, d\text{vol} \, \left(r^2 \bar{\omega} - (\bar{r} \cdot \bar{\omega}) \bar{r} \right) \qquad [\rho = \text{DENSITY}]$$

IN COMPONENT FORM, $(i, j, k) \leftrightarrow (x, y, z)$

$$L_i = \int \rho \, d\text{vol} \, \left(r^2 \omega_i - \sum_j r_j \omega_j r_i \right)$$

DEFINE $\delta_{ij} = \begin{cases} 1 & \text{IF } i = j \\ 0 & \text{IF } i \neq j \end{cases}$

$$L_i = \sum_j \omega_j \int \rho \, d\text{vol} \, \left(r^2 \delta_{ij} - r_i r_j \right) \equiv \sum_j I_{ij} \omega_j$$

WHERE $\underline{I_{ij} = \int \rho \, d\text{vol} \, \left(r^2 \delta_{ij} - r_i r_j \right) = \text{INERTIA TENSOR}}$

- IT IS A 3x3 MATRIX
- IT IS SYMMETRIC: $I_{ij} = I_{ji}$

  $\Rightarrow$ 6 INDEPENDENT QUANTITIES SUMMARIZED BY THE SYMBOL I_{ij}
- IF i LABELS A SYMMETRY AXIS, THEN

$$I_{ii} = \int \rho \, d\text{vol} \, (r^2 - r_i^2) = \int \rho \, d\text{vol} \, r^2 (1 - \cos^2\theta) = \int \rho \, d\text{vol} \, r^2 \sin^2\theta$$

$= I_i$ = ORDINARY <u>MOMENT OF INERTIA</u> ABOUT THAT AXIS

[THIS IS ACTUALLY TRUE WHETHER OR NOT THE AXIS IS A SYMMETRY AXIS.]

- THE CROSS TERMS I_{ij} $(i \neq j)$ ARE SOMETIMES CALLED THE <u>PRODUCTS OF INERTIA</u>.
- WE SOMETIMES WRITE $\bar{\bar{I}}$ WITH TWO BARS $\Rightarrow$ TENSOR

  THEN $\underline{\bar{L} = \bar{\bar{I}} \cdot \bar{\omega}}$ IN THIS NOTATION. (DYADIC NOTATION)

- THE SAME INERTIA TENSOR $\overline{\overline{I}}$ APPEARS IF WE EXAMINE THE KINETIC ENERGY (RELATIVE TO THE C.M.).

$$T = \frac{1}{2}\int \rho\, d\text{vol}\; v^2 = \frac{1}{2}\int \rho\, d\text{vol}\, (\bar\omega \times \bar r)\cdot(\bar\omega\times\bar r)$$
$$= \frac{1}{2}\int \rho\, d\text{vol}\,\left(\omega^2 r^2 - (\omega\cdot r)^2\right)$$
$$= \frac{1}{2}\sum_{ij} \omega_i \omega_j \int \rho\, d\text{vol}\,\left(\delta_{ij} r^2 - r_i r_j\right)$$
$$= \frac{1}{2}\sum_{ij} \omega_i I_{ij} \omega_j = \frac{1}{2}\,\bar\omega\cdot\overline{\overline{I}}\cdot\bar\omega$$

EULER'S EQUATIONS OF MOTION

[EULER IS PRONOUNCED 'OILER']

OUR ROTATIONAL EQUATION FOR RIGID BODY MOTION IS NOW

$$\bar N = \frac{d\bar L}{dt} = \frac{d}{dt}\left(\overline{\overline{I}}\cdot\bar\omega\right) = \frac{d\overline{\overline{I}}}{dt}\cdot\bar\omega + \overline{\overline{I}}\cdot\frac{d\bar\omega}{dt}$$

FOR ARBITRARY ROTATIONS, $\overline{\overline{I}}$ VARIES WITH TIME IN A MOST DISGUSTING MANNER, AND A SOLUTION SEEMS HOPELESS.

A GREAT SIMPLIFICATION (DUE TO L. EULER) IS TO GO TO AN ACCELERATED FRAME IN WHICH THE COORDINATE AXES ARE FIXED WITH RESPECT TO THE RIGID BODY. THESE AXES ARE CALLED THE <u>BODY AXES</u>; THE FRAME IS CALLED THE <u>BODY FRAME</u>.

CERTAINLY RELATIVE TO THESE AXES, $\overline{\overline{I}}$ IS CONSTANT IN TIME.

BUT THE BODY AXES ARE ROTATING WITH RESPECT TO OUR INERTIAL FRAME (THAT OF THE SO-CALLED <u>SPACE AXES</u>) WITH ANGULAR VELOCITY $\bar\omega$. HENCE

$$\frac{d\bar L}{dt} = \frac{\delta \bar L}{\delta t} + \bar\omega\times\bar L$$

[RECALL: $\frac{\delta}{\delta t}$ ≡ DERIVATIVE IN ROTATING FRAME]

AND $\frac{\delta\bar L}{\delta t} = \overline{\overline{I}}\cdot\frac{d\bar\omega}{dt}$ SINCE $\frac{\delta\overline{\overline{I}}}{\delta t} = 0$ [RECALL $\frac{\delta\bar\omega}{\delta t} = \frac{d\bar\omega}{dt}$]

HENCE $\underline{\bar N = \overline{\overline{I}}\cdot\dot{\bar\omega} + \bar\omega\times\overline{\overline{I}}\cdot\bar\omega}$

IS OUR 'SIMPLIFIED' EQUATION OF MOTION.

NATURE AND MATHEMATICS PERMIT A FURTHER SIMPLIFICATION.

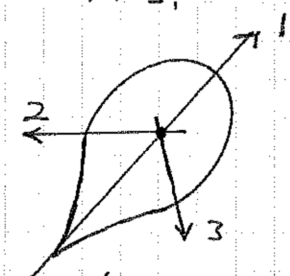
AS $\overline{\overline{I}}$ IS A SYMMETRIC TENSOR, THERE ALWAYS EXISTS A CHOICE OF COORDINATE AXES (IN THE BODY FRAME) SUCH THAT $\overline{\overline{I}}$ IS DIAGONAL.

i.e. $I_{ij} = 0$ IF $i \neq j$ $\Rightarrow I_{ij} = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix}$

THE SPECIAL SET OF AXES IS CALLED THE <u>PRINCIPAL AXES</u>. THE 3 MOMENTS OF INERTIA I_1, I_2, I_3 ARE CALLED THE <u>PRINCIPAL MOMENTS OF INERTIA</u>.

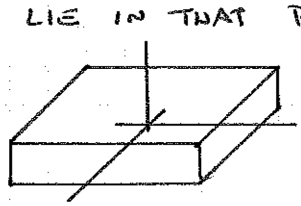
IN THE APPENDIX BELOW WE SKETCH A PROOF THAT THE PRINCIPAL AXES DO INDEED EXIST. FOR NOW, WE COMMENT ON SOME SPECIAL CASES WHICH GIVE IMPORTANT INSIGHT INTO THE CONCEPT OF PRINCIPAL AXES.

a) IF THE BODY IS <u>ROTATIONALLY SYMMETRIC</u> ABOUT AN AXIS, THEN THAT AXIS IS A PRINCIPAL AXIS.

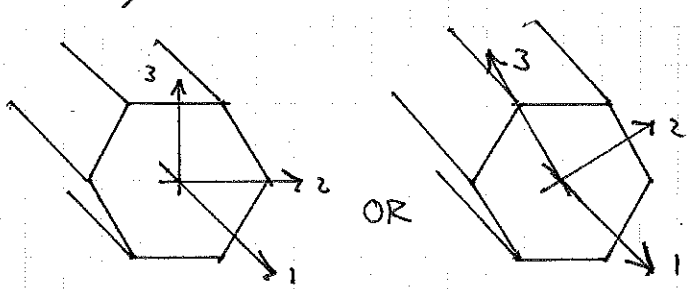


THE OTHER 2 PRINCIPAL AXES CAN BE ANY PAIR OF ORTHOGONAL AXES IN THE PLANE ⊥ TO THE SYMMETRY AXIS THRU THE C.M. SINCE THE CHOICE OF AXES 2 & 3 IS SOMEWHAT ARBITRARY, WE MUST HAVE $I_2 = I_3$. OF COURSE, $I_1 = I_2 = I_3$ OCCURS FOR A <u>SPHERICALLY SYMMETRIC</u> OBJECT. (BUT A CUBE ALSO HAS $I_1 = I_2 = I_3$, WHICH REMAINS TRUE NO MATTER WHAT AXES ARE CHOSEN!)

b) IF A BODY IS SYMMETRIC ABOUT A <u>PLANE</u>, TWO PRINCIPAL AXES LIE IN THAT PLANE.



A BOOK HAS ITS PRINCIPAL AXES PARALLEL TO ITS EDGES. IN GENERAL $I_1 \neq I_2 \neq I_3$



A HEXAGONAL PRISM HAS SEVERAL CHOICES OF AXES 2 & 3. HENCE $I_2 = I_3$ $(\neq I_1)$

NOTE THAT FOR $\overline{\omega}$ ALONG A PRINCIPAL AXIS, $\overline{L} = I\,\overline{\omega}$ HOLDS. FOR AN ARBITRARILY SHAPED OBJECT THERE ARE ALWAYS 3 DIRECTIONS FOR WHICH THIS SIMPLE FORM HOLDS (INSTANTANEOUSLY).

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IF WE USE THE PRINCIPAL AXES TO CALCULATE THE INERTIA TENSOR IN THE BODY FRAME, THEN THE EQUATIONS OF MOTION

$$\bar{N} = \bar{\bar{I}} \cdot \dot{\bar{\omega}} + \bar{\omega} \times \bar{\bar{I}} \cdot \bar{\omega}$$

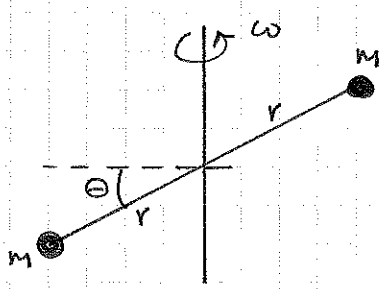
BECOME

$$\boxed{\begin{aligned} N_1 &= I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) \\ N_2 &= I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) \\ N_3 &= I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) \end{aligned}}$$

THE <u>EULER</u> <u>EQUATIONS</u>

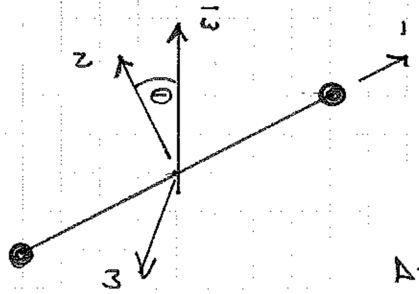
NOTE THAT WHILE $\bar{N}$ IS DEFINED IN THE LAB FRAME, ITS COMPONENTS ARE TAKEN ALONG THE INSTANTANEOUS DIRECTIONS OF THE PRINCIPAL AXES.

<u>EXAMPLE</u> PROBLEM 10, SET 1.



A DUMBELL IS FORCED TO ROTATE WITH CONSTANT ANGULAR VELOCITY ω ABOUT AN AXIS WHICH IS TILTED WITH RESPECT TO THE DUMBELL AXIS.

WHAT TORQUE IS REQUIRED TO MAINTAIN THIS MOTION?



THE PRINCIPAL AXES ARE EASILY SEEN TO BE AS SHOWN. ($\bar{\omega}$ IS IN THE 1-2 PLANE)

THEN $I_1 = 0, \quad I_2 = I_3 = 2mr^2$

ALSO $\omega = (\omega \sin\theta, \omega \cos\theta, \theta)$; $\dot{\bar{\omega}} = 0$

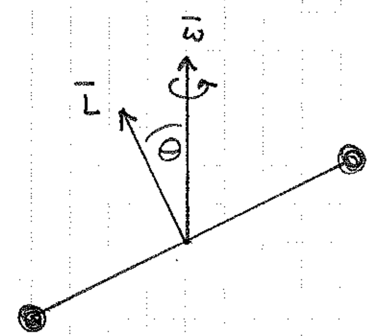
AT ONCE EULER TELLS US THAT

$$N_1 = N_2 = 0 \quad ; N_3 = 2mr^2 \omega^2 \sin\theta \cos\theta$$

IT IS INSTRUCTIVE TO LOOK AT $\bar{L} = \bar{\bar{I}} \cdot \bar{\omega}$

$$\Rightarrow L_1 = L_3 = 0 \quad , L_2 = 2mr^2 \omega \cos\theta$$

$\bar{L}$ IS NOT $\parallel$ TO $\bar{\omega}$!



SINCE $\bar{L} = L_2 \hat{2}$ ROTATES WITH THE BODY AXES

$$\frac{d\bar{L}}{dt} = \bar{\omega} \times \bar{L} = \bar{N} \Rightarrow N = \omega L \sin\theta = 2mr^2 \omega^2 \sin\theta \cos\theta$$

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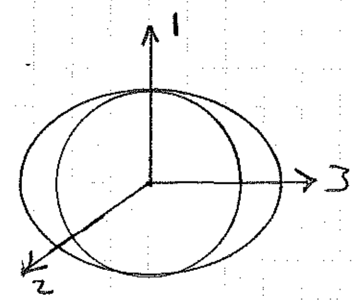
EXAMPLE FREE PRECESSION OF THE EARTH



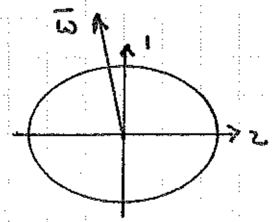
DUE TO CENTRIFUGAL FORCE, THE EARTH BULGES A BIT AT THE EQUATOR.

THUS $I_2 = I_3$ BUT $I_1 \neq I_2$

IF THE EARTH HAS UNIFORM DENSITY, A CALCULATION BASED ON THE OBSERVED R_{POLE} AND $R_{EQUATOR}$ YIELDS

$$\frac{I_1 - I_2}{I_2} = .00327 \equiv \epsilon$$

SUPPOSE THE AXIS OF ROTATION OF THE EARTH IS NOT QUITE THE PRINCIPAL AXIS 1. THEN EVEN IN THE ABSENCE OF ANY EXTERNAL TORQUE, A PRECESSION OF THE AXIS OF ROTATION OCCURS (WITH RESPECT TO THE FIXED STARS)



EULER TELLS US THAT IF $\bar{N} = 0$

$$\dot{\omega}_1 = 0$$

$$\left.\begin{aligned}\dot{\omega}_2 &= -\omega_1\omega_3 \frac{I_1 - I_2}{I_2} = -\epsilon\,\omega_1\omega_3 \\ \dot{\omega}_3 &= +\omega_1\omega_2 \frac{I_1 - I_2}{I_2} = +\epsilon\,\omega_1\omega_2\end{aligned}\right\} \text{USING } I_2 = I_3$$

THUS $\epsilon\omega_1$ IS A CONSTANT, AND THESE EQUATIONS CAN BE COMBINED INTO THE VECTORIAL EQUATION

$$\dot{\bar{\omega}} = \epsilon\,\omega_1\,\hat{1} \times \bar{\omega}$$

THAT IS, $\bar{\omega}$ PRECESSES ABOUT THE $\hat{1}$ AXIS WITH ANGULAR VELOCITY $\Omega = \epsilon\omega_1$. THE PERIOD IS $T = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega_1}\frac{1}{\epsilon} = \frac{1\text{ DAY}}{\epsilon} = \underline{306\text{ DAYS}}$

THE OBSERVED DEVIATION OF $\bar{\omega}$ FROM $\hat{1}$ AMOUNTS TO ONLY 10 METERS AT THE SURFACE OF THE EARTH. A PRECESSION PERIOD OF 420 DAYS IS ACTUALLY OBSERVED (ALONG WITH OTHER IRREGULAR CHANGES IN THE DIRECTION OF $\bar{\omega}$). IT IS PRESUMED THAT $420 \neq 306$ BECAUSE THE EARTH IS NOT RIGID. LONG AGO THE AXIS OF ROTATION WAS THOUSANDS OF MILES FROM WHERE IT IS NOW — POSSIBLE ONLY IF THE CRUST SLIPS RELATIVE TO THE INTERIOR OF THE EARTH!

IN OUR DISCUSSION WE HAVE BEEN VAGUE AS TO THE MOTION OF $\bar{\omega}$ AND $\hat{1}$ IN THE INERTIAL FRAME OF THE FIXED STARS. IN FACT, NEITHER IS CONSTANT, BUT RATHER IT IS ANGULAR MOMENTUM $\bar{L}$ WHICH IS INVARIANT. WE WILL PURSUE THE ARGUMENT IN LECTURE 18.

APPENDIX: DIAGONALIZATION OF A SYMMETRIC TENSOR

WE START WITH TENSOR $\overline{\overline{I}}$ WITH COMPONENTS ALONG SOME CHOICE OF AXES SUCH THAT I_{ij} IS NOT DIAGONAL, BUT $I_{ij} = I_{ji}$.

WE SEEK 3 UNIT VECTORS $\hat{e}_1$, $\hat{e}_2$ AND $\hat{e}_3$ SUCH THAT

$\overline{\overline{I}} \cdot \hat{e} = \lambda\ \hat{e}$ (λ_1 NOT NECESSARILY EQUAL λ_2, ETC.)

SUCH A VECTOR $\hat{e}$ IS CALLED AN EIGENVECTOR; λ IS THE EIGENVALUE.

IN COMPONENTS: $\sum_j I_{ij} \hat{e}_j = \lambda e_i = \lambda \sum_j \delta_{ij} e_j$

THESE EQUATIONS HAVE A SOLUTION ONLY IF

$$| I_{ij} - \lambda \delta_{ij} | = 0$$

SINCE I_{ij} IS A 3x3 MATRIX, THIS IS A CUBIC EQUATION IN λ.

$\therefore$ THERE ARE 3 ROOTS, 3 λ'S, AND CORRESPONDINGLY, 3 EIGENVECTORS $\hat{e}_1$, $\hat{e}_2$ AND $\hat{e}_3$.

a) THE EIGENVECTORS AND EIGENVALUES ARE REAL IF I_{ij} IS SYMMETRIC.

NOW $\sum_i e_i^* \sum_j I_{ij} e_j = \sum_{ij} I_{ij} e^*_i e_j = \sum_i e_i^* \lambda e_i = \lambda$

SUPPOSE $\hat{e}$ HAS IMAGINARY COMPONENTS. THEN

$$\lambda^* = \Big(\sum_{ij} I_{ij} e_i^* e_j\Big)^* = \sum_{ij} I_{ij} e_i e_j^* = \sum_{ij} I_{ji} e_j^* e_i = \lambda$$

$I_{ij} = I_{ji}$

THUS $\lambda^* = \lambda \Rightarrow$ λ IS REAL $\Rightarrow \hat{e}$ CAN BE TAKEN AS REAL.
(ALTHOUGH $\hat{e}\, e^{i\theta}$ IS ALSO AN EIGENVECTOR)
FINALLY, $\lambda = \sum_{ij} I_{ij} e_i e_j \Rightarrow \lambda \geq 0$

b) THE DIFFERENT $\hat{e}$'S ARE ORTHOGONAL IF $\lambda_1 \neq \lambda_2 \neq \lambda_3$

$\overline{\overline{I}} \cdot \hat{e}_1 = \lambda_1 \hat{e}_1 \Rightarrow \hat{e}_2 \cdot \overline{\overline{I}} \cdot \hat{e}_1 = \lambda_1 \hat{e}_2 \cdot \hat{e}_1$

BUT $\hat{e}_2 \cdot \overline{\overline{I}} = \lambda_2 \hat{e}_2 \Rightarrow \hat{e}_2 \cdot \overline{\overline{I}} \cdot \hat{e}_1 = \lambda_2\ \hat{e}_2 \cdot \hat{e}_1$

SO IF $\lambda_1 \neq \lambda_2$ THEN $\hat{e}_1 \cdot \hat{e}_2 = 0$ AS CLAIMED.

IF $\lambda_1 = \lambda_2$, THEN $\hat{e}_1 \cdot \hat{e}_2$ MAY NOT BE ZERO.

BUT THEN, ANY VECTOR $\alpha \hat{e}_1 + \beta \hat{e}_2$ IS ALSO AN EIGENVECTOR.
SO WE CAN CHOOSE TWO LINEAR COMBO'S WHICH ARE IN FACT ORTHOGONAL.

c) NOW CHOOSE COORDINATE AXES ALONG $\hat{e}_1, \hat{e}_2, \hat{e}_3$

THEN $\hat{e}_1 = (1,0,0) \quad \hat{e}_2 = (0,1,0) \quad \hat{e}_3 = (0,0,1)$

THEN $$\overline{\overline{I}} \cdot \hat{e}_1 = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11} \\ I_{12} \\ I_{13} \end{pmatrix} = \lambda_1 \hat{e}_1 = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

HENCE $I_{11} = \lambda_1$, $I_{12} = I_{13} = 0$ ETC.

THAT IS, $\overline{\overline{I}}$ IS DIAGONAL USING $\hat{e}_1, \hat{e}_2, \hat{e}_3$ AS AXES, AND THE EIGENVALUES (ROOTS OF A CUBIC EQUATION) ARE THE DIAGONAL ELEMENTS.

d) WE CAN ALSO CONSTRUCT THE TRANSFORMATION FROM ONE SET OF AXES TO ANOTHER.

I.e., SUPPOSE $\hat{x}_1, \hat{x}_2$ AND $\hat{x}_3$ ARE THE ORIGINAL AXES, AND WE KNOW THE COMPONENTS OF OUR EIGENVECTORS ALONG THESE AXES:
$$\hat{e}_i = \sum_j c_{ij} \hat{x}_j \qquad c_{ij} = \hat{e}_i \cdot \hat{x}_j = \text{\underline{DIRECTION} \underline{COSINE}}$$

THE INVERSE RELATION IS STRAIGHTFORWARD:

IF $\hat{x}_i = \sum_j d_{ij} \hat{e}_j$, THEN $d_{ij} = \hat{x}_i \cdot \hat{e}_j = c_{ji}$

OR $\hat{x}_i = \sum_j c_{ji} \hat{e}_j$ (THE <u>INVERSE</u> OF A <u>'ROTATION'</u> MATRIX IS ITS <u>TRANSPOSE</u>)

FOR AN ARBITRARY VECTOR: $\bar{r} = \sum_i r_i \hat{x}_i = \sum_i \sum_j r_i c_{ji} \hat{e}_j$

IF WE WISH TO WRITE $\bar{r} = \sum_i r_i' \hat{e}_i$ IN TERMS OF OUR NEW AXES,

THEN WE SEE THAT $r_i' = \sum_j c_{ij} r_j$

FOR WHAT IT'S WORTH, $|c_{ij}| = 1$, SINCE $c_{ij}^{-1} = c_{ji}$ ....

IN PRINCIPLE IT'S ALL FAIRLY STRAIGHTFORWARD. KEEPING THE AXES AND INDICES IN ORDER IS A PROBLEM IF YOU'RE NEW AT IT, BUT PERSEVERE!