

WAVE MOTION

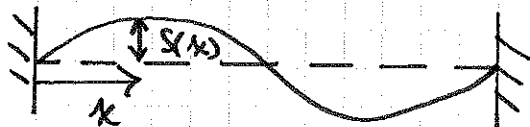
WE CLOSE THE COURSE WITH 4 LECTURES ON WAVE MOTION.

THUS FAR WE HAVE CONSIDERED THE MECHANICS OF PARTICLES AND RIGID BODIES. ANOTHER INTERESTING TYPE OF SYSTEM IS ONE IN WHICH THE NUMBER OF PARTICLES IS SO LARGE THAT THE MASS DISTRIBUTION IS ESSENTIALLY CONTINUOUS, BUT THE PARTICLES CAN MOVE RELATIVE TO ONE ANOTHER. SUCH SYSTEMS INCLUDE GASES, LIQUIDS, AND ELASTIC SOLIDS. THEIR STUDY MIGHT BE CALLED THE MECHANICS OF DEFORMABLE MEDIA

WAVE MOTION IS ONLY ONE PHENOMENON IN THIS RATHER LARGE FIELD, WHICH INCLUDES HYDRODYNAMICS, ELASTICITY, ETC. THE CLASSIC WORK ON WAVE MOTION IS 'THE THEORY OF SOUND' BY RAYLEIGH (DOVER PAPERBACK) WHICH WE WILL FOLLOW TO SOME EXTENT.

TRANSVERSE VIBRATIONS OF A STRING

AS THE SIMPLEST EXAMPLE, WE CONSIDER A STRING OF LENGTH l , MASS M , WHICH IS STRETCHED BETWEEN TWO POINTS WITH TENSION T . WE ARE INTERESTED IN MOTION IN WHICH THE STRING IS DISPLACED TRANSVERSELY TO ITS EQUILIBRIUM DIRECTION.



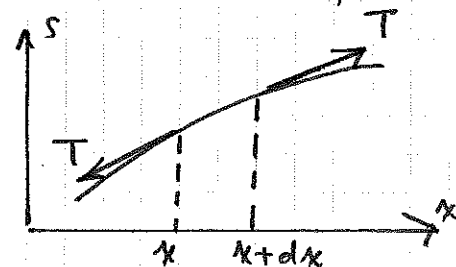
WE FIRST DERIVE THE EQUATION OF MOTION GOVERNING THE DISPLACEMENT $s(x, t)$.

THE LENGTH OF THE DISPLACED STRING IS GREATER THAN l . THE STRING MUST BE SOMEWHAT ELASTIC. HENCE THE TENSION IN THE DISPLACED STRING WILL BE GREATER THAN T . THE BASIC APPROXIMATION OF OUR ANALYSIS IS TO IGNORE THIS VARIATION IN T FOR SMALL DISPLACEMENTS. WE CONSIDER SOME ASPECTS OF POSSIBLE VARIATION IN T IN THE FOLLOWING LECTURES.

WE APPLY $F=ma$ TO AN ELEMENT OF STRING dx LONG, BETWEEN x AND $x+dx$.

LET $\rho = M/l =$ LINEAR DENSITY

$$\text{THEN } ma = \rho dx \frac{d^2 s}{dt^2}$$



Now $F =$ TRANSVERSE COMPONENT OF $(\vec{T}(x+dx) - \vec{T}(x))$

$$= T \sin \theta|_{x+dx} - T \sin \theta|_x$$

FOR SMALL ANGLES, $\sin \theta \approx \tan \theta \approx ds/dx$

$$\text{SO } F = T \left[\frac{ds(x+dx)}{dx} - \frac{ds(x)}{dx} \right] = T \frac{d^2s}{dx^2} dx$$

ALTOGETHER $\boxed{T \frac{d^2s}{dx^2} = \rho \frac{d^2s}{dt^2}}$ IS THE EQUATION OF MOTION.

IT IS VERY IMPORTANT TO UNDERSTAND THE PHYSICAL ORIGIN OF THE EQUATION.

THERE ARE 2 TYPES OF SOLUTIONS:

I. TRAVELLING WAVES $s = f(x-ct) + g(x+ct)$ IS A SOLUTION FOR ANY f, g , SO LONG AS

$$c = \sqrt{T/\rho} = \text{WAVE VELOCITY.}$$

WE WILL EXAMINE THESE SOLUTIONS IN DETAIL LATER.

II. STANDING WAVES. AN INTERESTING POSSIBILITY IS THAT A KIND OF FACTORIZED SOLUTION MIGHT EXIST:

$$s = f(x)g(t)$$

THIS METHOD OF SOLUTION IS CALLED SEPARATION OF VARIABLES, AS IS GENERALLY USEFUL FOR SOLVING LINEAR PARTIAL DIFFERENTIAL EQUATIONS. LET'S TRY IT:

$$T f'' g = \rho f \ddot{g}$$

$$\text{OR } \frac{T}{\rho} \frac{f''}{f} = \frac{\ddot{g}}{g} \leftarrow \text{FUNCTION OF } t \text{ ONLY}$$

\nwarrow FUNCTION OF x ONLY

THE ONLY POSSIBILITY IS THAT BOTH TERMS ARE EQUAL TO A CONSTANT; SAY $-\omega^2 =$ SEPARATION CONSTANT

$$\text{THEN } \ddot{g} = -\omega^2 g \quad f'' = -\omega^2 \frac{\rho}{T} f = -\frac{\omega^2}{c^2} f$$

NOTING $c = \sqrt{T/\rho} =$ WAVE VELOCITY.

HENCE OSCILLATORY SOLUTIONS ARE POSSIBLE:

$$g = A \cos \omega t + B \sin \omega t \quad f = C \cos \frac{\omega}{c} x + D \sin \frac{\omega}{c} x$$

$\omega =$ ANGULAR FREQUENCY OF OSCILLATION

$$\frac{2\pi c}{\omega} = \lambda = \text{WAVE LENGTH}$$

IT IS COMMON AND USEFUL TO INTRODUCE

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \equiv \underline{\text{WAVE NUMBER}} \quad (\omega = kc)$$

THEN $f = C \cos kx + D \sin kx$

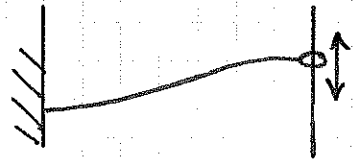
SO FAR A, B, C, D AND ω ARE ARBITRARY.

TO COMPLETE THE SOLUTION, SOME INITIAL CONDITIONS MUST BE SPECIFIED. BUT WE ALSO NOTE THAT THE PHYSICS OF SETTING UP THE STRETCHED STRING PUTS CONSTRAINTS ON THE POSSIBLE MOTION. THAT IS, THE MOTION AT THE ENDS OF THE STRING IS CONSTRAINED. THESE ARE THE SO-CALLED BOUNDARY CONDITIONS. THEIR NECESSARY EXISTENCE IS A QUALITATIVE DISTINCTION BETWEEN PARTIAL DIFFERENTIAL EQUATIONS AND ORDINARY DIFFERENTIAL EQUATIONS.

FOR A STRING, ONLY TWO KINDS OF BOUNDARY CONDITIONS ARE POSSIBLE - IF THE TENSION IS TO BE MAINTAINED,

a) THE STRING IS FIXED AT THE END: $S(0, t) = 0$ OR $S(l, t) = 0$

b) THE STRING IS ATTACHED TO A BEAD OR RING WHICH SLIDES ON A WIRE TRANSVERSE TO THE STRING. AN IDEALIZATION IS THAT THE RING IS MASSLESS AND FRICTIONLESS. THEN THE SLOPE OF THE STRING MUST ALWAYS BE ZERO AT THE RING. OTHERWISE $F = ma$ FOR THE RING $\Rightarrow a \rightarrow \infty$.



THE BOUNDARY CONDITION IS THUS $\frac{ds}{dx}(0, t) = 0$ OR $\frac{ds}{dx}(l, t) = 0$

c) OF COURSE ONE END COULD BE FREE AND THE OTHER END FIXED.

WE NOW EXAMINE HOW THE BOUNDARY CONDITIONS RESTRICT THE SOLUTIONS.

a) BOTH ENDS FIXED $f = C \cos kx + D \sin kx$

$$S(0, t) = 0 \Rightarrow f(0) = 0 \Rightarrow C = 0$$

$$S(l, t) = 0 \Rightarrow D \sin kl = 0 \Rightarrow k = \frac{n\pi}{l} \quad n = 1, 2, 3$$

$$\text{SO } \lambda = \frac{2\pi}{k} = \frac{2l}{n} = 2l, l, \frac{2l}{3}, \frac{l}{2}, \dots$$

$$\text{ALSO } \omega = kc = \frac{n\pi c}{l} \quad n = 1, 2, 3, \dots$$

HENCE ONLY CERTAIN FREQUENCIES AND WAVELENGTHS ARE POSSIBLE. IT IS NOT SURPRISING THAT THESE 'QUANTIZED' SOLUTIONS ARE CALLED THE NORMAL MODES.

ALTOGETHER
$$S(x,t) = \sum_n \sin \frac{n\pi x}{l} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right)$$

SINCE THE SUM OF SOLUTIONS TO A LINEAR EQUATION IS A SOLUTION ALSO. THE COEFFICIENTS A_n AND B_n MUST BE DETERMINED FROM THE INITIAL CONDITIONS.

NOTE THAT THE PHYSICS OF STRINGS HAS FORCED US TO DISCOVER THE FOURIER-SERIES EXPANSION!

b) BOTH ENDS FREE (ATTACHED TO RINGS)

$f = C \cos kx + D \sin kx$

$f' = -kC \sin kx + kD \cos kx$

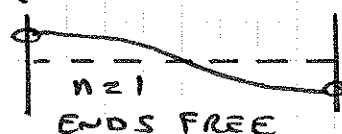
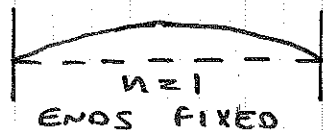
THE BOUNDARY CONDITIONS ARE $f'(0) = 0 \Rightarrow D = 0$

$f'(l) = 0 \Rightarrow \sin kl = 0$

SO AGAIN $k = \frac{n\pi}{l}$ $n = 0, 1, 2, 3$. (THE CASE $n=0$ IS SPECIAL, WHY?)

THE POSSIBLE FREQUENCIES ARE THE SAME AS BEFORE.

$$S(x,t) = \sum_n \cos \frac{n\pi x}{l} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right)$$



c) ONE END FIXED, ONE END FREE

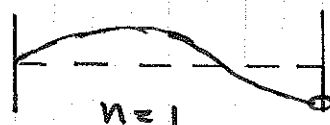
SAY $x=0$ IS FIXED $\Rightarrow f = D \sin kx$

$x=l$ FREE $\Rightarrow f'(l) = kD \cos kl = 0$

$\Rightarrow k = \frac{2n+1}{2} \frac{\pi}{l}$ $n = 0, 1, 2, \dots$


$\lambda = \frac{2\pi}{k} = \frac{4l}{2n+1} = 4l, \frac{4l}{3}, \frac{4l}{5} \dots$ $\omega = kc = \frac{(2n+1)\pi c}{2l}$

$$S(x,t) = \sum_n \sin \frac{(2n+1)\pi x}{2l} \left(A_n \cos \frac{(2n+1)\pi ct}{2l} + B_n \sin \frac{(2n+1)\pi ct}{2l} \right)$$



EXAMPLE. THE PLUCKED STRING

TO SHOW HOW THE SOLUTION IS COMPLETED, WE CONSIDER THE CASE OF A STRING PLUCKED AT $x=b$ INTO A TRIANGULAR FORM OF HEIGHT s_0 , THEN LET GO AT $t=0$. BOTH ENDS ARE FIXED



$$S(x,t) = \sum_n \sin \frac{n\pi x}{l} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right)$$

SO $S(x,0) = \sum_n A_n \sin \frac{n\pi x}{l}$ WHILE $\frac{dS}{dt} S(x,0) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$

SINCE THE STRING STARTS FROM REST AT $t=0$, ALL $B_n = 0$.

THE A_n ARE CALCULATED BY THE METHOD OF FOURIER:

MULTIPLY BY $\sin \frac{n\pi x}{l}$ AND INTEGRATE FROM 0 TO l

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l S(x,0) \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \int_0^b s_0 \frac{x}{b} \sin \frac{n\pi x}{l} + \frac{2}{l} \int_b^l s_0 \frac{l-x}{l-b} \sin \frac{n\pi x}{l} \\ &= \frac{2s_0 l^2}{\pi^2 b(l-b)} \frac{1}{n^2} \sin \frac{n\pi b}{l} \quad (\text{VIA INTEGRAL TABLE}) \end{aligned}$$

$$\text{AND } S(x,t) = \frac{2s_0 l^2}{\pi^2 b(l-b)} \sum_n \frac{1}{n^2} \sin \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

IF b IS A NODE OF THE n TH HARMONIC, $\sin \frac{n\pi b}{l} = 0$

AND THE HARMONIC IS SUPPRESSED.

TO GET THE 'PUREST' NOTE, PLUCK THE STRING AT ITS MIDPOINT. THEN THE 2ND, 4TH & ALL EVEN HARMONICS ARE SUPPRESSED. THE FIRST CONTAMINATING HARMONIC IS THE 3RD, WHICH HAS AMPLITUDE $1/9$ OF THE FUNDAMENTAL. (POWER $\sim 1/81$ OF FUNDAMENTAL)

BUT WHAT DOES THE VIBRATING STRING LOOK LIKE TO THE EYE? FOURIER'S METHOD TELLS US WHAT THE VIBRATION SOUNDS LIKE TO THE EAR

EXAMPLE THE SHARPLY STRUCK STRING

SUPPOSE INSTEAD THAT AT $t=0$ AN IMPULSE OF STRENGTH I IS APPLIED TO THE STRING AT A POINT b .

THE INITIAL CONDITION IS THEN $s(x,0) = 0$, AND $\dot{s}(x,0) = 0$ EVERYWHERE EXCEPT AT $x=b$, WHERE \dot{s} IS VERY LARGE DUE TO THE IMPULSE. REMEMBER THAT AN IMPULSE CAUSES A CHANGE IN MOMENTUM.

$$I = \int_{b-\epsilon}^{b+\epsilon} \rho dx \dot{s} \Rightarrow \dot{s}(x,0) = \frac{I}{\rho} \delta(x-b)$$

WE USE THE FAMOUS DIRAC DELTA FUNCTION

WHICH OBEYS $\delta(x-b) = 0$ IF $x \neq b$

$$\int \delta(x-b) dx = 1$$

AND SO $\int f(x) \delta(x-b) dx = f(b)$

TO SATISFY THE INITIAL CONDITION $s(x,0) = 0$, WE SEE THAT

$$s(x,t) = \sum_n B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$$

THEN $\dot{s}(x,0) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l} = \frac{I}{\rho} \delta(x-b)$

IT IS EASY TO USE FOURIER'S METHOD (NO INTEGRAL TABLE NEEDED!)

$$\frac{n\pi c}{l} B_n = \frac{2}{l} \int_0^l \frac{I}{\rho} \delta(x-b) \sin \frac{n\pi x}{l} = \frac{2I}{\rho l} \sin \frac{n\pi b}{l}$$

SO $s(x,t) = \frac{2I}{\pi \rho c} \sum_n \frac{1}{n} \sin \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$

SINCE THE TERMS GO AS $1/n$ THE SERIES CONVERGES MORE SLOWLY THAN IN THE CASE OF A PLUCKED STRING. IF $b=l/2$ THE 3RD HARMONIC CONTAMINATES THE FIRST WITH A RELATIVE AMPLITUDE OF $1/3$, AND THE NOTE IS NOT VERY PURE!

ARBITRARY DRIVING FORCES

SUPPOSE THE STRING IS SUBJECT TO AN EXTERNAL, TRANSVERSE FORCE $F(x,t)$. THE EQUATION OF MOTION IS THEN

$$\rho \ddot{s} = T s'' + F \quad \text{or} \quad \ddot{s} - c^2 s'' = \frac{F}{\rho}$$

FOR THE CASE OF FIXED ENDS, WE STILL HOPE FOR A SOLUTION OF THE FORM

$$s(x,t) = \sum_n \phi_n(t) \sin \frac{n\pi x}{l}$$

THE TIME DEPENDENCE OF s WILL FOLLOW THAT OF THE DRIVING FORCE, BUT THE DECOMPOSITION INTO SPATIAL MODES $\sin n\pi x/l$ REMAINS UNCHANGED.

PLUGGING IN:
$$\sum_n \left[\ddot{\phi}_n + \left(\frac{n\pi c}{l}\right)^2 \phi_n \right] \sin \frac{n\pi x}{l} = \frac{F(x,t)}{\rho}$$

BY FOURIER'S METHOD

$$\ddot{\phi}_n + \left(\frac{n\pi c}{l}\right)^2 \phi_n = \frac{2}{l\rho} \int_0^l F(x,t) \sin \frac{n\pi x}{l} dx \equiv \frac{F_n(t)}{\rho}$$

AND THE PROBLEM IS REDUCED TO AN INFINITE SERIES OF ONE-DIMENSIONAL DRIVEN OSCILLATOR PROBLEMS, FOR WHICH WE KNOW LOTS OF TRICKS:

a) $F_n(t)$ IS SIMPLE HARMONIC WITH FREQUENCY Ω

$$F_n(t) = F_n \cos \Omega t$$

$$\text{THEN } \phi_n = \frac{F_n \cos \Omega t}{\rho \left[\left(\frac{n\pi c}{l}\right)^2 - \Omega^2 \right]}$$

b) $F_n(t)$ IS PERIODIC WITH PERIOD $T = 2\pi/\omega$

$$\text{THEN WE CAN ANALYZE } \frac{F_n(t)}{\rho} = \frac{A_0}{2} + \sum_m A_m \cos m\omega t + B_m \sin m\omega t$$

$$\text{LEADING TO } \phi_n(t) = \frac{A_0}{2 \left(\frac{n\pi c}{l}\right)^2} + \sum_m \frac{A_m \cos m\omega t + B_m \sin m\omega t}{\left(\frac{n\pi c}{l}\right)^2 - m^2 \omega^2}$$

c) $F_n(t)$ IS ARBITRARY. WE CAN USE GREEN'S METHOD (P144)

$$\phi_n(t) = \frac{1}{\rho \frac{n\pi c}{l}} \int_{-\infty}^t F_n(t') \sin \frac{n\pi c}{l} (t-t') dt'$$

EXAMPLE. A SIMPLE HARMONIC DRIVING FORCE IS APPLIED AT THE POINT $x=b$:

$$F(x,t) = F \delta(x-b) \cos \Omega t$$

WE CAN USE METHOD a). THE FOURIER ANALYSIS OF $F(x,t)$

QUICKLY YIELDS $F_n(t) = \frac{2F}{l} \frac{\sin \frac{n\pi b}{l}}{l} \cos \Omega t$

$$\text{SO } S(x,t) = \frac{2F}{pl} \sum_n \frac{\sin \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \cos \Omega t}{\left(\frac{n\pi c}{l}\right)^2 - \Omega^2}$$

WHILE THE TIME DEPENDENCE OF THE VIBRATION IS JUST $\cos \Omega t$, WE DON'T KNOW DIRECTLY WHAT THE SHAPE OF THE VIBRATION IS — UNLESS WE CAN SUM THE SERIES.

AMAZINGLY, THIS CAN BE DONE!

THAT IS, WE CAN SOLVE THE PROBLEM BY ANOTHER METHOD, WHICH IS IN ITSELF VERY INSTRUCTIVE.

WE DIVIDE THE STRING INTO TWO PIECES $[0,b]$, AND $[b,l]$, SOLVE EACH PIECE SEPARATELY, AND THEN MATCH SOLUTIONS AT THE BOUNDARY, $x=b$. THE DRIVING FORCE WILL APPEAR ONLY AS A BOUNDARY CONDITION.

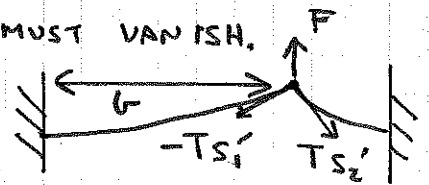
LET $S_1(x,t)$ = SOLUTION ON $[0,b]$, S_2 = SOLUTION ON $[b,l]$

THE BOUNDARY CONDITIONS ARE:

$$S_1(0,t) = 0, \quad S_2(l,t) = 0, \quad S_1(b,t) = S_2(b,t)$$

AND THAT THE SUM OF THE FORCES AT $x=b$ MUST VANISH.

$$T [S_2'(b,t) - S_1'(b,t)] + F \cos \Omega t = 0$$



OF COURSE WE EXPECT OSCILLATORY SOLUTIONS $\sim \cos \Omega t$

$$S_1(0,t) = 0 \Rightarrow S_1 = A_1 \sin k_1 x \cos \Omega t$$

$$S_2(l,t) = 0 \Rightarrow S_2 = A_2 \sin k_2 (l-x) \cos \Omega t$$

THE WAVE EQUATION RELATES k TO Ω :

$$c^2 S'' = \ddot{S} \Rightarrow k^2 c^2 = \Omega^2 \quad \text{OR} \quad k = \Omega/c$$

$$S_1 = A_1 \sin \frac{\Omega x}{c} \cos \Omega t$$

$$S_2 = A_2 \sin \frac{\Omega (l-x)}{c} \cos \Omega t$$

WE NOW APPLY THE MATCHING CONDITIONS AT $x = b$

$$A_1 \sin \frac{\Omega b}{c} = A_2 \sin \frac{\Omega}{c} (l-b)$$

$$\text{AND } T \left(A_1 \frac{\Omega}{c} \cos \frac{\Omega b}{c} + A_2 \frac{\Omega}{c} \cos \frac{\Omega}{c} (l-b) \right) = F$$

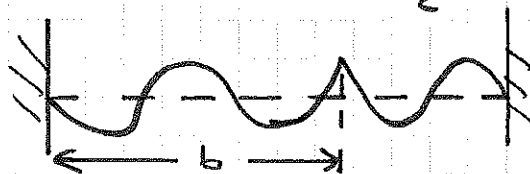
$$\text{SO } A_1 \left(\cos \frac{\Omega b}{c} + \cos \frac{\Omega}{c} (l-b) \frac{\sin \frac{\Omega b}{c}}{\sin \frac{\Omega}{c} (l-b)} \right) = \frac{Fc}{T\Omega}$$

$$A_1 = \frac{Fc}{T\Omega} \frac{\sin \frac{\Omega}{c} (l-b)}{\sin \frac{\Omega l}{c}}$$

$$A_2 = \frac{Fc}{T\Omega} \frac{\sin \frac{\Omega b}{c}}{\sin \frac{\Omega l}{c}}$$

$$S_1(x,t) = \frac{Fc}{T\Omega} \frac{\sin \frac{\Omega}{c} (l-b)}{\sin \frac{\Omega l}{c}} \frac{\sin \frac{\Omega x}{c}}{\sin \frac{\Omega l}{c}} \cos \Omega t$$

$$S_2(x,t) = \frac{Fc}{T\Omega} \frac{\sin \frac{\Omega b}{c}}{\sin \frac{\Omega l}{c}} \frac{\sin \frac{\Omega}{c} (l-x)}{\sin \frac{\Omega l}{c}} \cos \Omega t$$



WE MAY NOTE AN INTERESTING SYMMETRY: THE MOTION AT x DUE TO A FORCE AT b IS THE SAME AS THE MOTION AT b WERE THE FORCE APPLIED AT x INSTEAD.

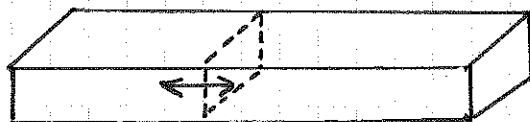
ALSO, NOTE THAT IF $\Omega = \frac{n\pi c}{l} =$ A NATURAL FREQUENCY, THEN $S_1, S_2 \rightarrow \infty$. THIS IS THE RESONANCE PHENOMENON AGAIN. IN REAL LIFE A DAMPING TERM (AIR RESISTANCE, INELASTICITY OF THE STRING ...) KEEPS THINGS FINITE.

$$\ddot{S} = c^2 S'' + \underbrace{\beta S'}_{\text{DAMPING}} + \gamma S \dots$$

LONGITUDINAL VIBRATIONS OF A BAR OR SPRING

ANOTHER SITUATION LEADING TO WAVE MOTION IS THE VIBRATION OF AN ELASTIC SOLID.

FOR NOW WE WILL CONSIDER ONLY ONE DIMENSIONAL VIBRATIONS; IN PARTICULAR THOSE WHICH ARE LONGITUDINAL - FOR WHICH ENTIRE PLANES OF MATTER VIBRATE ONLY \perp TO THE PLANE. WE ALSO SUPPOSE THE VIBRATING PLANES ARE PARALLEL TO THE SURFACE OF THE BAR.



THE BASIC BEHAVIOR OF AN ELASTIC SOLID IS THAT IF STRETCHED, THE SOLID DEVELOPS A TENSION FORCE PROPORTIONAL TO THE AMOUNT OF STRETCH (HOOKE'S LAW). WE PUT THIS INTO A FORM ADVOCATED BY YOUNG:

$$\frac{\text{FORCE}}{\text{AREA}} = Y \frac{\Delta L}{L} \quad \left[Y = \frac{\text{STRESS}}{\text{STRAIN}} = \frac{F/A}{\Delta L/L} \right]$$

WHERE $Y = \text{YOUNG'S MODULUS}$, AND IS FOUND IN HANDBOOKS ON PROPERTIES OF MATERIALS.

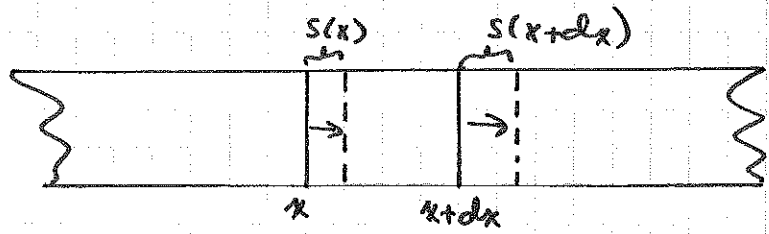
FOR A BAR OF LENGTH L , CROSS SECTIONAL AREA A , THE SPRING CONSTANT IS JUST

$$K = \frac{YA}{L}$$

WE NOTE THAT THE SPRING CONSTANT DEPENDS ON THE DIRECTION OF ΔL , AND THAT THE INTERNAL STRESS MAY NOT BE EXACTLY ALONG ΔL K IS REALLY A TENSOR! WE WILL CONSIDER ONLY MOTION PARALLEL TO A 'PRINCIPAL AXIS' - SO A SCALAR RELATION WILL SUFFICE.

WE WILL ALSO CONSIDER ONLY BARS OF CONSTANT CROSS SECTION, SO

$$\text{FORCE} = AY \frac{\Delta L}{L} = \lambda \frac{\Delta L}{L}$$



TO DERIVE THE EQUATION OF MOTION, CONSIDER A SLICE THRU THE BAR WHICH IS dx LONG, BETWEEN x AND $x+dx$.

LET $S(x, t) =$ DISPLACEMENT OF THE PLANE ORIGINALLY AT x

THEN $F = ma$ BECOMES $ma = \rho dx \ddot{S}(x)$ ($\rho =$ LINEAR DENSITY)

WITH $F = F(x+dx) - F(x)$

$$= \lambda \left(\frac{S(x+dx) - S(x)}{dx} \right) - \lambda \left(\frac{S(x) - S(x-dx)}{dx} \right)$$

$$= \lambda (S'(x+dx) - S'(x)) = \lambda S''(x) dx$$

SO $\boxed{\ddot{S} = \frac{\lambda}{\rho} S''}$ IS THE WAVE EQUATION

THE WAVE VELOCITY IS $c = \sqrt{\lambda/\rho}$

SO ALL OUR STUDIES OF THE SOLUTION TO THE WAVE EQUATION APPLY AT ONCE.

FOR A BAR, A COMMON BOUNDARY CONDITION IS THAT THE ENDS ARE FREE:

$$S'(0, t) = 0 = S'(l, t) \iff \text{NO FORCE AT THE FREE ENDS}$$

LEADING TO $S(x, t) = \sum_n \cos \frac{n\pi x}{l} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right)$

ETC, ETC.