Ph 205 Solutions to Midterm 2

① Central Force $\bar{F} = -\frac{\alpha}{r^2} e^{-\beta r}\,\hat{r}$

This can in principle be derived from a potential $V(r) = -\int_\infty^r \bar{F}\cdot d\bar{r} = \alpha \int_\infty^r \frac{e^{-\beta r}}{r^2}\,dr$

The key is that we do not actually have to do the integral!

The energy, in polar coordinates is $E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r) = \text{const}$

Angular momentum is also conserved: $L = m r^2 \dot{\theta} = \text{const}$

$$\therefore \quad E = \underbrace{\frac{1}{2} m \dot{r}^2}_{T_{eff}} + \underbrace{\frac{L^2}{2 m r^2} + V(r)}_{V_{eff}}$$

The condition for an equilibrium circular orbit is $V'_{eff}(r_0) = 0$

The condition for stability of small oscillations is $V''_{eff}(r_0) \geq 0$

$$V'_{eff} = -\frac{L^2}{m r^3} + \frac{\alpha}{r^2} e^{-\beta r}$$

$$V''_{eff} = \frac{3L^2}{m r^4} - \frac{2\alpha}{r^3} e^{-\beta r} - \frac{\alpha \beta e^{-\beta r}}{r^2}$$

$$V'_{eff}(r_0) = 0 \quad \Rightarrow \quad \frac{L^2}{m r_0} = \alpha e^{-\beta r_0}$$

The equilibrium angular velocity Ω satisfies $L = m r_0^2 \Omega$

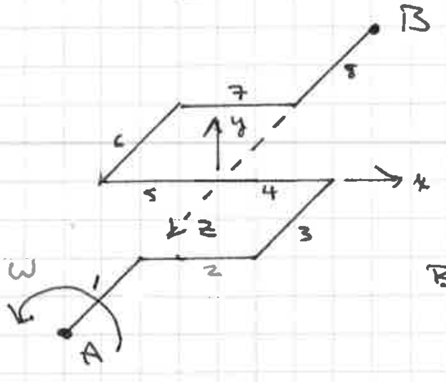
so $\boxed{m \Omega^2 r_0 = \frac{\alpha e^{-\beta r_0}}{r_0^2}}$ (which also follows from $F = ma$ at once)

$$V''_{eff}(r_0) = 3m\Omega^2 - m\Omega^2 r_0 \left(\frac{2}{r_0} + \beta\right) = m\Omega^2 (1 - r_0 \beta)$$

$$\omega^2_{osc} = \frac{V''_{eff}(r_0)}{m} = \Omega^2 (1 - r_0\beta) \quad \Rightarrow \quad \boxed{\omega = \Omega \sqrt{1 - r_0 \beta}}$$

Clearly for stability we must have $\boxed{r_0 < \frac{1}{\beta}}$

② The crankshaft. We can use several methods. But since the CM is at rest, $\bar{F}_A + \bar{F}_B = 0$ in all methods



a) Rotating Frame.

The crankshaft is at rest, but we experience the centrifugal force. By the symmetry, $\sum \bar{F}_{centrifugal} = 0$

But the torque due to the centrifugal force does not vanish!

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ROD #	$\bar{F}_{centric}$	PT OF APPLICATION	$\bar{N}$ = TORQUE
1	0		0
2	$m\frac{a}{2}\omega^2\hat{x}$	$(\frac{a}{2}, 0, a)$	$m\frac{a^2}{2}\omega^2\hat{y}$
3	$m a \omega^2 \hat{x}$	$(a, 0, a/2)$	$m a^2/2\, \omega^2 \hat{y}$
4	$m\, a/_2\, \omega^2 \hat{x}$	$(a/_2, 0, 0)$	0
5	$-m\, a/2\, \omega^2 \hat{x}$	$(-a/_2, 0, 0)$	0
6	$-m a \omega^2 \hat{x}$	$(-a, 0, -a/2$	$m a^2/2\, \omega^2 \hat{y}$
7	$-m\, a/_2\, \omega^2 \hat{x}$	$(-a/_2, 0, -a)$	$m a^2/_2\, \omega^2 \hat{y}$
8	0		0

$$\bar{N}_c = 2 m a^2 \omega^2 \hat{y}$$

NOW $\bar{r}_A \times \bar{F}_A + \bar{r}_B \times \bar{F}_B + \bar{N}_c = 0$

$$2\bar{r}_A \times \bar{F}_A = -2 m a^2 \omega^2 \hat{y} \qquad \bar{r}_A = 2a\hat{z}$$

$$\Rightarrow \boxed{\bar{F}_A = -\bar{F}_B = -\frac{m a \omega^2}{2}\hat{x}}$$

b) STAY IN LAB FRAME AND USE $\bar{N} = \frac{d\bar{L}}{dt}$

WHERE $N = \bar{r}_A \times \bar{F}_A + \bar{r}_B \times \bar{F}_B = 4a\hat{z} \times \bar{F}_A$

RECALL $\bar{L} = \bar{R} \times \bar{P}_{cm} + \bar{L}_{REL\ TO\ CM}$

SO WE CAN CALCULATE THE PIECES OF $\bar{L}$ DUE TO EACH ROD

ROD	$\bar{R} \times \bar{P}$	$\bar{L}_{REL}$
1	0	0
2	$(\frac{a}{2}, 0, a) \times (0, \frac{M\omega a}{2}, 0) = (-\frac{M\omega a^2}{2}, 0, \frac{M\omega a^2}{4})$	$\frac{1}{12} M a^2 \omega \hat{z}$
3	$(a, 0, a/_2) \times (0, M\omega a, 0) = (-M\omega a^2/_2, 0, M\omega a^2)$	0
4	$(a/_2, 0, 0) \times (0, M\omega a/2, 0) = (0, 0, M\omega a^2/4)$	$\frac{1}{12} M a^2 \omega \hat{z}$
5	$(-a/_2, 0, 0) \times (0, -M\omega a/_2, 0) = (0, 0, M\omega a^2/4)$	$\frac{1}{12} M a^2 \omega \hat{z}$
6	$(-a, 0, -\frac{a}{2}) \times (0, -M\omega a, 0) = (-M\omega a^2/_2, 0, M\omega a^2)$	0
7	$(-a/_2, 0, a) \times (0, -M\omega a/_2, 0) = (-M\omega^2 a/_2, 0, M\omega a^2/(4))$	$\frac{1}{12} M a^2 \omega \hat{z}$
8	0	0

$$\bar{L} = -2 M \omega a^2 \hat{x} + 10/_3\, M \omega a^2 \hat{z}$$

$$\frac{d\bar{L}}{dt} = -2 m \omega a^2 \frac{d\hat{x}}{dt} \qquad \text{AND} \qquad \frac{d\hat{x}}{dt} = \bar{\omega} \times \hat{x} = \omega \hat{y}$$

SO $\bar{N} = -2 m \omega^2 a^2 \hat{y} = 4a\hat{z} \times \bar{F}_A \Rightarrow \boxed{\bar{F}_A = -\frac{m a \omega^2}{2}\hat{x} = -\bar{F}_B}$

OR WE COULD USE $\bar{L} = \bar{\bar{I}} \cdot \bar{\omega} = \bar{\bar{I}} \cdot \hat{z}\omega = I_{xz}\omega\hat{x} + I_{yz}\omega\hat{y} + I_{zz}\omega\hat{z}$

$I_{yz} = 0$ SINCE THE CRANK IS IN THE X-Z PLANE

I_{zz} = MOMENT ABOUT Z AXIS $= 4 \cdot \frac{1}{3} M a^2 + 2 \cdot M a^2 = \frac{10}{3} M a^2$

$$I_{ij} = \int \rho\, dvol \left(r^2 \delta_{ij} - r_i r_j\right)$$

$I_{xz} = -\int \rho\, xz\, dvol$. WE WORK THIS OUT FOR THE RODS ONE BY ONE

$$I_{xz}(1) = 0 = I_{xz}(8)$$
$$I_{xz}(2) = -\frac{m}{a}\cdot a\int_0^a x\,dx = -\tfrac{1}{2}ma^2$$
$$I_{xz}(3) = -\frac{m}{a}\cdot a\int_0^a z\,dz = -\tfrac{1}{2}ma^2$$
$$I_{xz}(4) = 0 = I_{xz}(5)$$
$$I_{xz}(6) = -\frac{m}{a}(-a)\int_{-a}^0 z\,dz = -\tfrac{1}{2}ma^2$$
$$I_{xz}(7) = -\frac{m}{a}(-a)\int_{-a}^0 x\,dx = -\tfrac{1}{2}ma^2$$

SO $I_{xz} = -2ma^2$

AND $\bar{L} = -2ma^2\omega\,\hat{x} + \frac{10}{3}ma^2\omega\,\hat{z}$ AS BEFORE.

THERE IS NO NEED TO DIAGONALIZE THE INERTIAL TENSOR.

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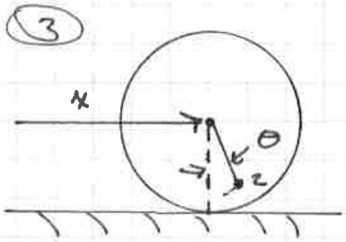

I CHOOSE x OF C.M. OF HOOP
θ OF MASS 2 TO THE VERTICAL
AS COORDS

2 DEGREES OF FREEDOM $\Rightarrow$ 2 MODES

BUT IF WE THINK A BIT, WE NOTE THAT THE HOOP CAN ROLL WITHOUT SLIPPING AT CONSTANT VELOCITY, WHILE $\theta = 0$ ALWAYS $\Rightarrow$ NO OSCILLATION.

$\therefore$ WE MAY EXPECT THAT $\omega = 0$ FOR ONE MODE!

IN THE OTHER MODE WE MIGHT EXPECT x AND θ TO OSCILLATE OPPOSITELY....

$L = T + V$ $\quad T_1 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I\omega_{Hoop}^2 = m_1\dot{x}^2 \quad T_2 = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$

$$x_2 = x + l\sin\theta \qquad \dot{x}_2 = \dot{x} + l\cos\theta\,\dot{\theta}$$
$$y_2 = r - l\cos\theta \qquad \dot{y}_2 = l\sin\theta\,\dot{\theta}$$

$$T_2 = \tfrac{1}{2}m_2(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\,l\cos\theta\,\dot{\theta}$$

$$V = m_2 g\, y_2 = m_2 g\,(r - l\cos\theta)$$

$$L = \tfrac{1}{2}(2m_1+m_2)\dot{x}^2 + m_2\dot{x}\,l\cos\theta\,\dot{\theta} + \tfrac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 g\,l\cos\theta$$

$\frac{\partial L}{\partial \dot{x}} = (2m_1+m_2)\dot{x} + m_2 l\cos\theta\,\dot{\theta}$ = CONST SINCE $\frac{\partial L}{\partial x} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2\dot{x}\,l\cos\theta + m_2 l^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_2\ddot{x}\,l\cos\theta - m_2\dot{x}\,l\sin\theta\,\dot{\theta} + m_2 l^2\ddot{\theta} = \frac{\partial L}{\partial \theta} = -m_2\dot{x}\,l\sin\theta\,\dot{\theta} - m_2 g\,l\sin\theta$$

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FROM THE X EQUATION $\ddot{x} = \frac{-m_2}{2m_1+m_2} l \cos\theta\, \ddot{\theta} + \frac{m_2}{2m_1+m_2} l \sin\theta\, \dot{\theta}^2$

$$\text{SO} \quad \ddot{\theta}\left(1 - \frac{m_2}{2m_1+m_2}\cos^2\theta\right) + \frac{m_2}{2m_1+m_2}\sin\theta\cos\theta\,\dot{\theta}^2 = -\frac{g}{l}\sin\theta$$

THIS CAN CERTAINLY BE SATISFIED BY $\underline{\theta = 0 \Rightarrow \dot{x} = \text{CONST} \; ; \; x = v t}$

IF θ NOT CONSTANT, WE MAKE THE SMALL ANGLE APPROXIMATION

$$\frac{2m_1}{2m_1+m_2}\ddot{\theta} \approx -\frac{g}{l}\theta$$

$$\Rightarrow \quad \underline{\theta = A\cos\omega t} \quad \text{WHERE} \quad \boxed{\omega = \sqrt{\frac{2m_1+m_2}{2m_1}\frac{g}{l}}}$$

$$\dot{x} \approx -\frac{m_2}{2m_1+m_2} l\dot{\theta} \Rightarrow x = \frac{-m_2}{2m_1+m_2} l\theta = \underline{\frac{-m_2}{2m_1+m_2} lA\cos\omega t}$$

$$\underline{x_{MAX} = +\frac{m_2}{2m_1+m_2} l\,\theta_{MAX}}$$

X AND θ ARE INDEED 180° OUT OF PHASE AS EXPECTED

NOTE THAT IF THE C.M. OF THE SYSTEM DID NOT MOVE HORIZONTALLY, THEN WE EXPECT

$$m_1 x + m_2(x + l\theta) = 0$$

$$\text{OR} \quad x = -\frac{m_2}{m_1+m_2} l\theta$$

BUT THE C.M. IS NOT NECESSARILY AT REST BECAUSE OF THE CONSTRAINT FORCE!