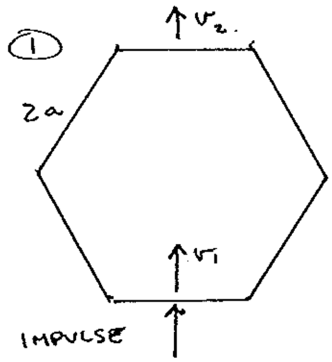
Ph 205 SET 4

DUE TUESDAY OCT 17, 1989

MAXIMUM RECORDED SCORE = 80 POINTS

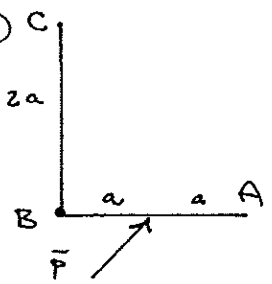
①



A REGULAR HEXAGON HAS EDGES OF LENGTH $2a$. THE EDGES ARE RIGID RODS OF MASS M WHICH ARE JOINED TOGETHER WITH FRICTIONLESS PIVOTS. IF A BLOW IS STRUCK PERPENDICULAR TO THE MIDPOINT OF THE LOWER EDGE, SHOW THAT

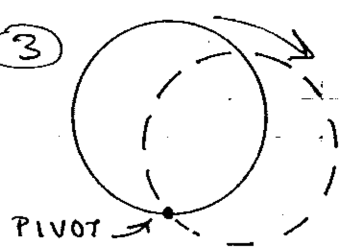
$$V_2 = \frac{1}{10} V_1 \quad \text{JUST AFTERWARDS.}$$

②



RODS AB AND BC BOTH HAVE MASS M AND LENGTH $2a$. THEY ARE JOINED AT B BY A FRICTIONLESS PIVOT. INITIALLY THE RODS ARE AT REST, WITH ANGLE $ABC = 90°$. A BLOW IS STRUCK TO THE MIDPOINT OF ROD AB SUCH THAT THE SYSTEM MOVES AS A RIGID BODY IN THE NEXT INSTANT (I.E. ANGLE ABC REMAINS 90° JUST AFTER THE IMPULSE). SHOW THAT IN THIS CASE THE IMPULSE MAKES A 45° ANGLE TO ROD AB, AND THAT $V_A = \sqrt{13}\, V_C$ JUST AFTER THE IMPULSE.

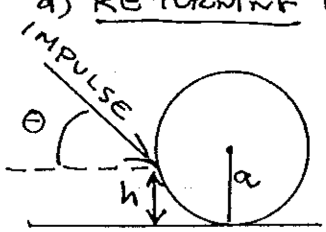
③



A UNIFORM DISC OF MASS M ROTATES FREELY IN A VERTICAL PLANE ABOUT A POINT ON ITS CIRCUMFERENCE. INITIALLY THE DISC IS BALANCED ABOVE THE PIVOT — THEN IT FALLS. SHOW THAT THE FORCE ON THE PIVOT IS $\frac{\sqrt{17}}{3} Mg$ WHEN THE DISC HAS ROTATED BY 90°, AND $11/3\, Mg$ WHEN IT HAS ROTATED BY 180° (SO IT IS DIRECTLY BELOW THE PIVOT BUT STILL MOVING). IT IS INSTRUCTIVE TO EXAMINE THE FORCE COMPONENTS ALONG BOTH x-y AND r-θ AXES.

④ BILLIARDS

a) RETURNING BALL

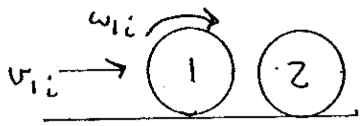


IF THE CUE BALL IS STRUCK WITH THE CUE HORIZONTAL, THE BALL CAN NEVER RETURN. BUT IF THE CUE IS TILTED, IT IS POSSIBLE.

SUPPOSE THE IMPULSE AND THE CENTER OF THE BALL LIE IN A VERTICAL PLANE. SHOW THAT THE ANGLE OF THE CUE MUST OBEY

$$\tan\theta \geq \frac{1}{\sqrt{\frac{2a}{h} - 1}} \quad \text{FOR THE BALL TO RETURN}$$

(4) b) <u>FOLLOW SHOTS</u> & <u>DRAW SHOTS</u> (SEE B&O p188)



THE CUE BALL STRIKES ANOTHER BALL OF THE SAME MASS AND RADIUS SUCH THAT THE LINE OF CENTERS OF THE TWO BALLS IS ALONG THE DIRECTION OF THE INITIAL VELOCITY. ω_{1i} IS NOT NECESSARILY EQUAL TO V_{1i}/a AT THE MOMENT OF IMPACT.
(TAKE $\bar{\omega}_{1i}$ AS $\perp$ TO $\bar{V}_{1i}$ AND $\parallel$ TO THE TABLE).

ASSUME NO FRICTION BETWEEN THE BALLS DURING IMPACT, AND THAT THE COLLISION IS ELASTIC.

SHOW THAT WHEN BALL 2 FINALLY ROLLS WITHOUT SLIPPING THAT $V_{2f} = \frac{5}{7} V_{1i}$ (OF COURSE $V_{2i} = 0$)

ALSO SHOW THAT WHEN BALL 1 FINALLY ROLLS WITHOUT SLIPPING THAT $V_{1f} = \frac{2}{7} a\,\omega_{1i}$

SO BALL 1 FOLLOWS BALL 2 UNLESS $\omega_{1i} \leq 0$

c) <u>ENGLISH</u> (<u>OPTIONAL</u>) IF THE CUE IS NOT HORIZONTAL, AND IS NOT POINTING AT THE CENTER OF THE BALL, THE BALL ACQUIRES '<u>ENGLISH</u>' AND DOES NOT MOVE IN A STRAIGHT LINE. WE WISH TO CALCULATE THE PATH OF THE C.M.

IF THE BALL IS NOT TO MOVE IN A STRAIGHT LINE, THERE MUST BE A COMPONENT OF FRICTION $\perp$ TO $\bar{V}_{CM}$. THIS MEANS THERE MUST BE SOME ROTATION ALONG AN AXIS $\parallel$ TO $\bar{V}_{C.M.}$ WE CALL THIS 'ENGLISH'.

THE CONCEPT OF 'ENGLISH' MAY ALSO INCLUDE ROTATION ABOUT AN AXIS $\perp$ TO THE TABLE. HOWEVER THIS ROTATION DOES NOT INFLUENCE THE MOTION <u>BETWEEN</u> COLLISIONS. IF THE AREA OF CONTACT OF THE BALL WITH THE TABLE IS SMALL, THERE IS NO FRICTIONAL TORQUE ABOUT THE VERTICAL AXIS, SO ω_z NEVER CHANGES – AND $\bar{V}_{CM}$ IS COMPLETELY UNAFFECTED.

IN THIS PROBLEM THE SPHERE HAS ROTATION $\bar{\omega}(t)$ ABOUT AN INSTANTANEOUS AXIS WHOSE DIRECTION VARIES WITH TIME. TYPICALLY THIS IS BIG TROUBLE, BUT FOR A SPHERE THE MOMENT OF INERTIA IS THE SAME ABOUT ANY AXIS THRU THE C.M. HENCE $\bar{L} = I\bar{\omega}$ WITH $I = \frac{2}{5} m a^2$ NO MATTER WHERE $\bar{\omega}$ POINTS.

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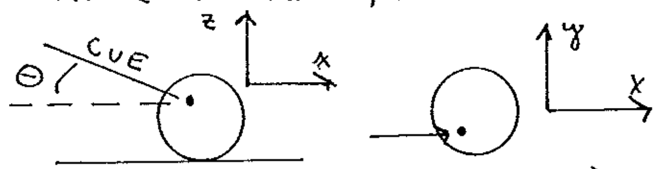
THE ANALYSIS CONSISTS OF 2 PARTS:

a) ESTABLISHING THE INITIAL MOTION CAUSED BY THE CUE

b) CALCULATING THE SUBSEQUENT MOTION — CHANGES IN WHICH ARE CAUSED BY THE FRICTION OF THE TABLE AGAINST THE BALL.

TO FIND THE INITIAL MOTION, CHOOSE A COORDINATE SYSTEM WITH z VERTICAL, AND THE CUE IN THE $x-z$ PLANE.



SIDE VIEW TOP VIEW

THE ORIGIN IS AT THE CENTER OF THE SPHERE.

LET (X, Y, Z) = COORD OF CONTACT PT. OF THE CUE.

$$X^2 + Y^2 + Z^2 = a^2$$

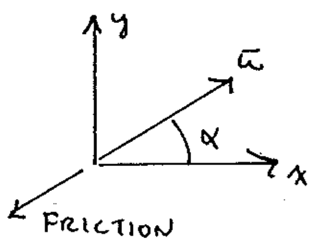
IF P IS THE MAGNITUDE OF THE IMPULSE, THEN $\bar{P} = (P\cos\theta, 0, -P\sin\theta)$

CALCULATE V_{x_0}, ω_{x_0} AND ω_{y_0}

TO EXAMINE THE SUBSEQUENT MOTION, USE A TRICK.

DEFINE $\bar{u}$ = VELOCITY OF THE POINT ON THE BALL INSTANTANEOUSLY IN CONTACT WITH THE TABLE.

($\bar{u} = 0$ FOR ROLLING WITHOUT SLIPPING).



$$u_x = u\cos\alpha$$
$$u_y = u\sin\alpha$$

THE FORCE OF SLIDING FRICTION IS THEN

$$\bar{f} = -\mu m g\,\hat{u}$$

THE EQUATIONS OF MOTION ARE $m\dot{\bar{v}} = \bar{f}$

AND $I\dot{\bar{\omega}} = \bar{r} \times \bar{f}$

WHERE $\bar{r} = (0, 0, -a)$

USE THESE TO CALCULATE $\dot{u}_x$ AND $\dot{u}_y$ AND THEN $\dot{u}$ AND $\dot{\alpha}$

BE VERY CAREFUL RELATING $\bar{u}$ TO $\bar{v}$ AND $\bar{\omega}$.

SHOW THAT THE BALL BEGINS TO ROLL WITHOUT SLIPPING AT TIME $t = \frac{2}{7}\frac{u_0}{\mu g}$

AND THAT $\dot{\alpha} = $ CONSTANT ALWAYS.

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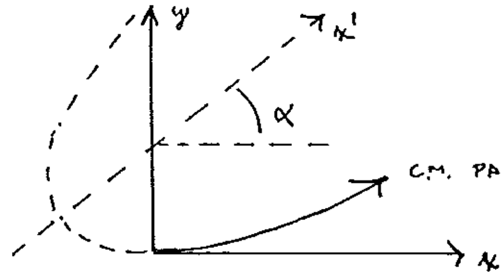
HENCE IN A COORD SYSTEM WITH AXIS x' ALONG DIRECTION $\vec{\omega}$

$$v_{y'} = \text{CONSTANT} \qquad \text{AND} \qquad \frac{d}{dt} v_{x'} = \text{CONSTANT}$$

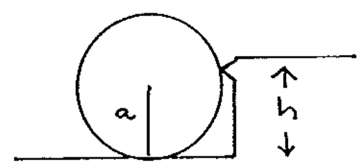
SO THE PATH OF THE C.M. IS A PARABOLA WHOSE AXIS MAKES ANGLE α TO THE x-AXIS



VERIFY THAT
$$\tan\alpha = \frac{-\frac{5}{2}\frac{Y}{a}\sin\theta}{\left(1 - \frac{5}{2}\frac{Z}{a}\right)\cos\theta - \frac{5}{2}\frac{X}{a}\sin\theta}$$

IF EITHER $\sin\theta = 0$ OR $Y = 0$ THEN $\alpha = 0$ AND THE PATH REDUCES TO A STRAIGHT LINE.

(5) A BALL OF RADIUS a ROLLS WITH C.M. VELOCITY v_0, AND ANGULAR VELOCITY ω_0 UNTIL IT COLLIDES WITH A 'BUMPER' OF HEIGHT h. DURING THE COLLISION THERE IS NO SLIPPING AT THE POINT OF IMPACT.

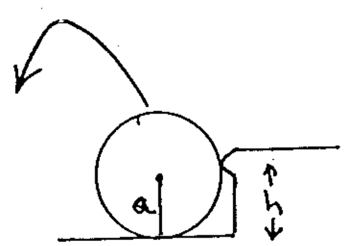


a) IF $h < a$, SHOW THAT THE BALL CAN 'JUMP' UP ONTO THE STEP IF

$$v_0 \geq \frac{a}{a-h}\sqrt{\frac{14gh}{5}} \qquad \text{AND} \qquad \omega_0 = 0$$

$$\text{or} \qquad v_0 \geq \frac{a}{7a-5h}\sqrt{70gh} \qquad \text{WHEN} \qquad \omega_0 = v_0/a$$

IF $\omega_0 = \frac{v_0}{a}$ AND $a < h < \frac{7}{5}a$, THE BALL WILL STILL JUMP UPWARDS AND REACH A HEIGHT $> h + a$ IF THE ABOVE CONDITION ON v_0 IS SATISFIED. HOWEVER IT BOUNCES BACKWARDS, FALLING BACK ONTO THE TABLE.

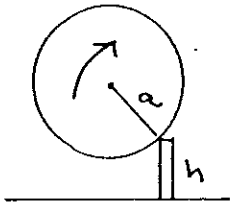


b) SUPPOSE $h < a$ IF v_0 AND h ARE GREAT ENOUGH THE BALL WILL LOSE CONTACT WITH THE CUSHION (OR STEP) AND FLY INTO THE AIR. SUPPOSE $v_0 = v_{0\,MIN}$ FOUND ABOVE.

SHOW THAT THE BALL FLIES IF $h \geq \frac{7}{17}a$

FOR BOTH CASES $\omega_0 = \frac{v_0}{a}$ AND $\omega_0 = 0$.

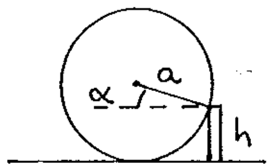
c) (OPTIONAL)



SUPPOSE THE STEP IS JUST A POST OF HEIGHT h. IF THE BALL FLIES UP, WILL IT HIT THE POST ON THE WAY BACK DOWN?

SHOW THAT THE BALL WILL HIT THE POST AT TIME t GIVEN BY

$$\frac{1}{4} g^2 t^2 - g v (\cos \alpha) t + v^2 - a g \sin \alpha = 0$$



WHERE v = VELOCITY OF C.M. OF BALL JUST AFTER HITTING THE POST.

IF THIS EQUATION HAS REAL, POSITIVE ROOTS THE BALL HITS THE POST ON THE WAY DOWN. THIS IS CLAIMED TO BE TRUE.

⑥ RELATIVISTIC ROCKET WE WON'T USE MUCH RELATIVITY IN Ph 205, BUT YOU MAY WANT TO TRY THIS PROBLEM TO KEEP LIMBER. IN THE NON-RELATIVISTIC ROCKET PROBLEM WE USED CONSERVATION OF MASS AND MOMENTUM TO DERIVE THE EQUATION OF MOTION. IN RELATIVITY WE REPLACE CONSERVATION OF MASS BY CONSERVATION OF TOTAL ENERGY INCLUDING THE 'REST ENERGY'

TRY IT YOURSELF. IF YOU FIND YOU NEED HELP, WE OUTLINE A POSSIBLE ANALYSIS BELOW....

WE CAN USE A TYPICAL RELATIVITY TRICK. FIRST CALCULATE THE EQUATION IN A FRAME IN WHICH THE ROCKET IS INITIALLY AT REST. THEN TRANSFORM TO THE LAB FRAME.

IN OUR SPECIAL FRAME IN WHICH THE ROCKET IS INITIALLY AT REST, LET M_1 = INITIAL ROCKET MASS. AFTER A SHORT TIME THE ROCKET HAS REST MASS M_2 AND VELOCITY dv IN THIS FRAME, AS A RESULT OF SPEWING OUT EXHAUST OF REST MASS m AT VELOCITY u RELATIVE TO THE ROCKET.

BUT $m \neq M_1 - M_2$ IN GENERAL

WRITE DOWN CONSERVATION OF ENERGY AND MOMENTUM IN THIS FRAME TO SHOW

$$dv = u \frac{dM}{M}$$ WHERE $u > 0$ BY DEFINITION AND $dM = M_1 - M_2$

WE ARE APPROXIMATING $M \sim M_1 \sim M_2$ AND IGNORING TERMS IN $(dv)^2$ SUCH AS $\dfrac{1}{\sqrt{1 - \frac{(dv)^2}{c^2}}}$

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IN THE LAB FRAME THE ROCKET HAS INITIAL VELOCITY V AND FINAL VELOCITY $V + dV$. USE THE VELOCITY TRANSFORMATION TO SHOW $\gamma^2 dV = dv$ WHERE $\gamma = \dfrac{1}{\sqrt{1 - V^2/c^2}}$

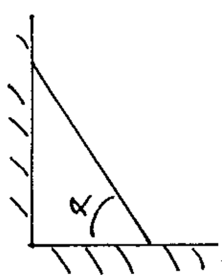
IN FACT, $\gamma^2 dV$ IS A <u>RELATIVISTIC INVARIANT</u> WHICH YOU COULD VERIFY BY SUPPOSING dv IS FINITE. (YOU COULD NO LONGER IGNORE TERMS LIKE $\dfrac{1}{\sqrt{1-(dv)^2/c^2}}$).

FURTHERMORE, $\gamma^2 dV$ IS A PERFECT DIFFERENTIAL:

$$\gamma^2 \frac{dV}{c} = d\left[\tanh^{-1}\left(\frac{V}{c}\right)\right]$$

SHOW $$\frac{V}{c} = \frac{\left(\frac{M_0}{M}\right)^{\frac{2w}{c}} - 1}{\left(\frac{M_0}{M}\right)^{\frac{2w}{c}} + 1}$$ IF $V = 0$ WHEN $M = M_0$

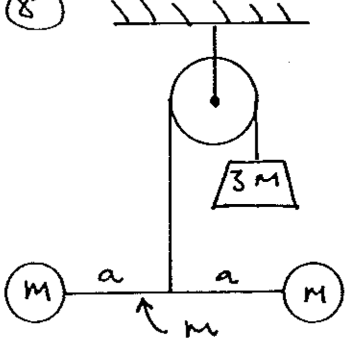
(7)



A LADDER STANDS AGAINST A FRICTIONLESS WALL WITH ITS FEET ON A FRICTIONLESS FLOOR.
IF IT IS LET GO WITH INITIAL ANGLE α_0, SHOW THAT THE LADDER LOSES CONTACT WITH THE WALL WHEN

$$\sin\alpha = \tfrac{2}{3}\sin\alpha_0$$

(8)

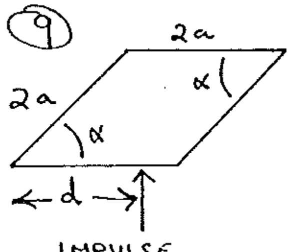


A ROD OF MASS M, LENGTH $2a$ HAS MASSES M ATTACHED AT BOTH ENDS. A STRING TIED TO THE MIDDLE OF THE ROD GOES OVER A MASSLESS, FRICTIONLESS PULLEY AND IS TIED TO A MASS $3M$.

AT $t = 0$ ONE OF THE MASSES M FALLS OFF THE END OF THE ROD. SHOW THAT THE ROD ROTATES WITH INITIAL $\ddot{\theta} = \frac{18}{17}\, g/a$

AND THAT THE TENSION IN THE STRING IS $\frac{30}{17} mg$ INITIALLY.

(9)



OUR FAMOUS RHOMBUS LIES ON A HORIZONTAL PLANE AT REST. INITIALLY THE ACUTE ANGLE IS α. AN IMPULSE IS APPLIED PERP. TO A ROD AT DISTANCE d FROM THE ACUTE CORNER.

WHAT SHOULD d BE SUCH THAT THE RHOMBUS ROTATES AND TRANSLATES BUT DOES NOT CHANGE SHAPE? ASSUME FRICTIONLESS PIVOTS AT THE CORNERS.

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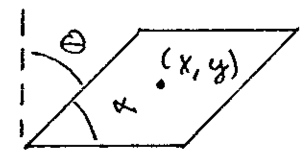
ANS: $d = a(1-\cos\alpha)$

WHAT SHOULD d BE SUCH THAT THE RHOMBUS TRANSLATES AND DEFORMS BUT DOES NOT ROTATE (I.E. A DIAGONAL DOES NOT ROTATE)? ANS: $d = a(1+\cos\alpha)$ ⇒ BLOW LINES UP WITH THE C.M.!

THIS PROBLEM SHOWS THE IMPORTANCE OF A GOOD CHOICE OF COORDINATES IN LAGRANGE'S METHOD. THERE ARE 4 DEGREES OF FREEDOM – 2 OF TRANSLATION, ROTATION, AND DEFORMATION. WE WANT COORDS SUCH THAT EACH COORD. CORRESPONDS TO ONLY ONE OF THESE MOTIONS. THAT IS, q_j APPEARS IN THE ENERGY OF ONLY ONE SUCH MOTION. IF SO, THEN WE SIMPLY ARRANGE THAT THE GENERALISED IMPULSE OF TYPE j VANISHES, AND THE IMPULSE DOES NOT AFFECT COORDINATE j.

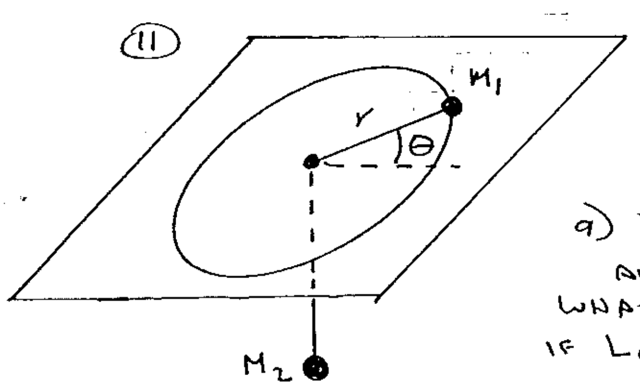
A <u>BAD</u> CHOICE OF COORDINATES IS (x, y, α, θ) AS SHOWN



BECAUSE ROTATION DEPENDS ON BOTH α AND θ....

(10) PROBLEMS 24 & 25, P199 B&O. YOU MAY WISH TO FOLLOW THE GENERAL CASE THRU 2 BOUNCES TO GET STARTED. (SEE P. 8)

(11)



MASS M_1 IS CONNECTED TO MASS M_2 BY A STRING WHICH PASSES THRU A HOLE IN THE TABLE. THE TABLE IS HORIZONTAL AND FRICTIONLESS.

a) DERIVE THE EQUATIONS OF MOTION AND CONSTRUCT THE EFFECTIVE POTENTIAL. WHAT IS THE EQUILIBRIUM RADIUS r_0, IF L_0 = INITIAL ANGULAR MOMENTUM?

SHOW THAT THE FREQUENCY OF SMALL OSCILLATIONS ABOUT r_0 IS

$$\omega = \sqrt{\frac{3 M_2 g}{(M_1+M_2) r_0}}$$

b) SUPPOSE THAT M_1 IS INITIALLY AT r_0 MOVING WITH CONSTANT VELOCITY IN A CIRCLE. THEN SOMEONE THROWS DIRT ON THE TABLE CAUSING A VERY SMALL COEFFICIENT OF SLIDING FRICTION μ. HOW FAR DOES M_1 TRAVEL BEFORE COMING TO REST? HOW LONG DOES IT TAKE TO COME TO REST, ASSUMING THE ORBIT REMAINS NEARLY CIRCULAR AT ALL TIMES?

5-11 SUPER-BALL BOUNCES

The bizarre behavior observed in bounces of the Wham-O Super-Ball[1] can be predicted from the rigid-body equations of motion.[2] The Super-Ball is a hard spherical rubber ball. The bounces of a Super-Ball on a hard surface are almost elastic (i.e., energy-conserving) and essentially nonslip at the point of contact. As an idealization, we shall assume kinetic-energy conservation during a bounce. We shall also neglect gravity in our calculations, though its inclusion does not change the principal results.

We begin with an analysis of a single bounce from the floor. We denote the initial components of the CM velocity by V_x^0 and V_y^0 and the initial spin of the ball about the z axis through the CM by ω_z^0, as pictured in Fig. 5-18. The frictional force f_x and the normal force f_y act on the ball only for a very short time duration, Δt. We can determine the changes in the velocities ΔV_x, ΔV_y from the linear-impulse equation (5-103), and the change in spin from the angular-impulse equation (5-105). We obtain

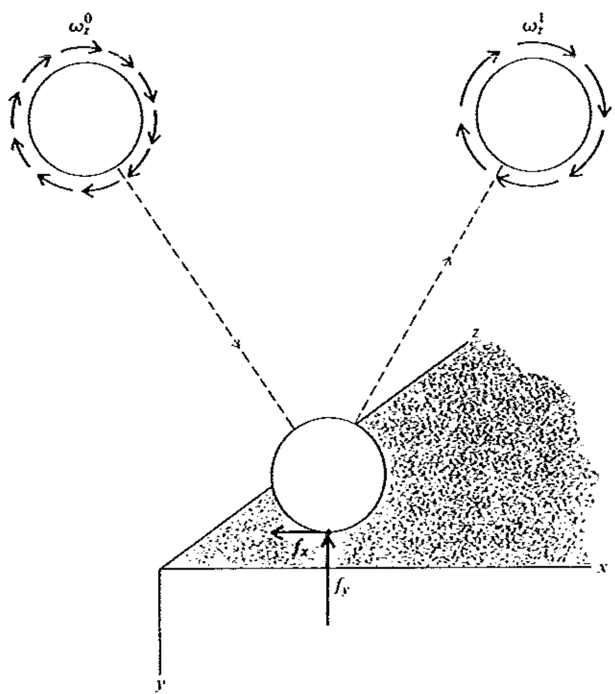
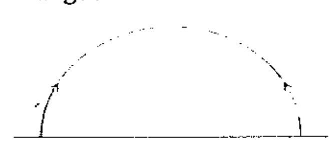


FIGURE 5-18 Super-Ball bounce from a hard surface.

[1] Registered trademark of Wham-O Corporation, San Gabriel, Calif.
[2] R. L. Garwin, *Amer. J. Phys.*, 37: 88 (1969).

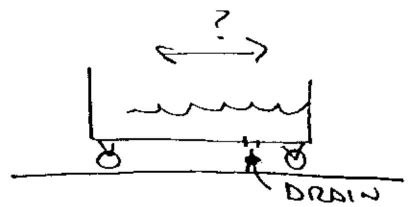
24. One of the Super-Ball examples discussed in the text concerned a ball dropped straight down with spin. Discuss the subsequent motion through several bounces.

25. Under what conditions will a Super-Ball bounce back and forth as illustrated? How does the spin change?



CHALLENGE PROBLEM : NOT FOR CREDIT

DISCUSS THE MOTION OF A TANK CAR INITIALLY AT REST ONCE A DRAIN ON THE BOTTOM IS OPENED. THE CAR ROLLS WITHOUT FRICTION ON A HORIZONTAL PLANE. THE WATER FLOWS OUT OF THE DRAIN VERTICALLY IN THE REST FRAME OF THE CAR.



THE PROBLEM IS NON TRIVIAL IF THE DRAIN IS OFF CENTER.

A RATHER COMPLETE DISCUSSION CAN BE GIVEN WITH NO EQUATIONS, INDEPENDENT OF THE EXACT FORM OF $\dot{M}(t)$ = RATE OF FLOW OF WATER OUT OF THE TANK.

EVENTUALLY YOU MAY WISH TO CONSIDER SOME SPECIFIC EXAMPLES, SUCH AS $\dot{M}(t) =$ CONSTANT (NOT TOO REALISTIC) OR $\dot{M}(t)$ SUCH THAT THE WATER LEAVING THE TANK TAKES ALL THE POTENTIAL ENERGY OF THE WATER WITH IT...