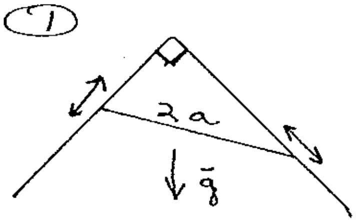
Ph 205 SET 5

DUE: THURS OCT 25, 1990
MAXIMUM RECORDED SCORE = 60 POINTS

PROBLEMS 1, 4 AND 6 WILL NOT BE GRADED

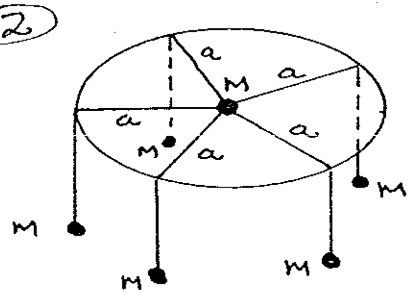
1



A UNIFORM ROD OF LENGTH $2a$ HAS ITS ENDS CONSTRAINED TO MOVE ON TWO WIRES WHICH MAKE 45° ANGLES TO THE VERTICAL AS SHOWN.

SHOW THAT THE LENGTH OF A SIMPLE PENDULUM WITH THE SAME FREQUENCY OF OSCILLATION IS $\ell = \frac{4a}{3}$

2

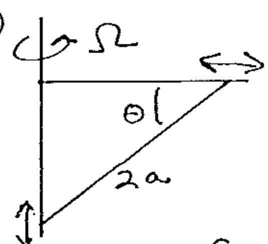


A MASS M IS CONNECTED VIA n STRINGS TO n OTHER MASSES M. THE STRINGS HANG OVER THE EDGE OF A CIRCULAR TABLE OF RADIUS a. THE STRINGS ARE CONSTRAINED TO PASS OVER FIXED POINTS ON THE EDGE OF THE TABLE, WHICH POINTS ARE UNIFORMLY SPACED. THERE IS NO FRICTION.

SHOW THAT THE FREQUENCY OF SMALL OSCILLATIONS ABOUT EQUILIBRIUM IS $\omega = \sqrt{\frac{ng}{a(n+2)}}$ FOR $n > 2$.

FOR $n = 2$, $\omega = \sqrt{\frac{2g}{a}}$ FOR OSCILLATIONS TRANSVERSE TO THE STRING, BUT $\omega = 0$ FOR 'OSCILLATIONS' ALONG THE STRING.

3



THE GIZMO OF PROBLEM 1 IS REARRANGED SO THAT ONE WIRE IS VERTICAL, THE OTHER IS HORIZONTAL. THEN IT IS MADE TO ROTATE WITH CONSTANT ANGULAR VELOCITY Ω ABOUT THE VERTICAL WIRE.

CONSTRUCT THE EFFECTIVE POTENTIAL AND SHOW THAT FOR STEADY MOTION THE EQUILIBRIUM ANGLES ARE

$$\theta_0 = \pi/2 \quad \text{AND} \quad \sin\theta_0 = \frac{3g}{4a\Omega^2}$$

ALSO SHOW THAT THE FREQUENCY OF SMALL OSCILLATIONS IS

$$\omega = \sqrt{\frac{3g}{4a}}\sqrt{1 - \frac{4a\Omega^2}{3g}} \quad \text{OR} \quad \omega = \Omega\sqrt{1 - \left(\frac{3g}{4a\Omega^2}\right)^2}$$

DEPENDING ON WHICH EQUILIBRIUM IS STABLE.

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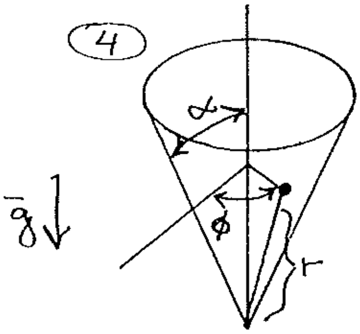
(4) 

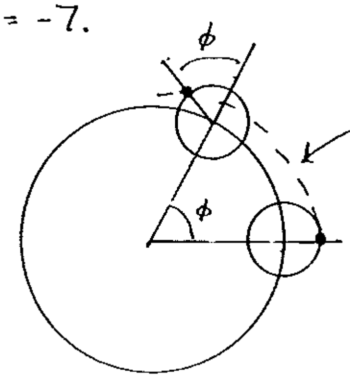
A CONE OF HALF ANGLE α HAS ITS AXIS VERTICAL AND TIP DOWNWARDS. A PARTICLE SLIDES FREELY ON THE INNER SURFACE OF THE CONE. CONSTRUCT AND SKETCH THE EFFECTIVE POTENTIAL. WHAT IS THE EQUILIBRIUM RADIUS r_0 AS A FUNCTION OF $L_0 \equiv$ ANGULAR MOMENTUM?
WHAT IS THE EQUILIBRIUM ANGULAR VELOCITY $\Omega \equiv \dot{\phi}$?
FIND THE FREQUENCY OF SMALL OSCILLATIONS ABOUT EQUILIBRIUM. SKETCH THE ORBITS FOR SMALL DEPARTURES FROM EQUILIBRIUM FOR CONES OF $\sin \alpha = \frac{1}{2}$ $\frac{1}{\sqrt{3}}$ AND $\frac{1}{\sqrt{3}}$.

(5) CONSIDER THE CENTRAL-FORCE PROBLEM WITH A POTENTIAL OF THE FORM $V(r) = -\frac{C}{r^\lambda}$ $(\lambda \neq 0)$

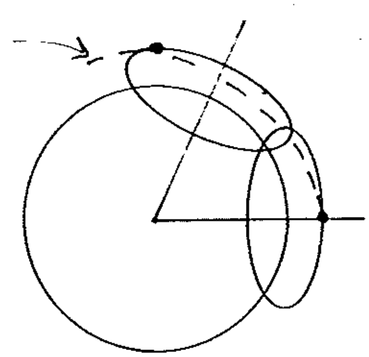
SHOW THAT CIRCULAR ORBITS ARE STABLE ONLY FOR $\lambda < 2$, AND THAT ORBITS WHICH DEPART ONLY SLIGHTLY FROM CIRCULARITY CAN BE WRITTEN

$$r = r_0 (1 + \epsilon \cos \omega t)$$
$$\theta = \Omega t - \frac{2\epsilon}{\sqrt{2-\lambda}} \sin \omega t$$

WHERE $\omega = \sqrt{2-\lambda}\,\Omega$, AND Ω IS THE ANGULAR VELOCITY OF THE CIRCULAR ORBIT. THESE ORBITS ARE SIMPLE CLOSED CURVES ONLY IF $\sqrt{2-\lambda}$ IS AN INTEGER. SKETCH THE ORBITS, RELATIVE TO THE FORCE CENTER FOR $\lambda = +1$ (GRAVITY), $\lambda = -2$ (SPRINGS) AND $\lambda = -7$.



- - - - = ACTUAL ORBIT



THE GREEK EPICYCLE

THE NEWTONIAN ORBIT AS A SMALL OSCILLATION ABOUT A CIRCLE (C.F. P 98a, LECTURE 9)

(6) a) SHOW THAT THE CLOSED-CIRCULAR-ORBIT PERIOD OF A PEBBLE AROUND A BOULDER IS (ROUGHLY) THE SAME AS THE PERIOD OF A LOW-ALTITUDE SATELLITE AROUND THE EARTH. (I.E., SHOW THAT THE CLOSED-ORBIT-PERIOD DEPENDS ONLY ON THE DENSITY OF THE LARGE BODY.)

(6) b) <u>SATELLITE PARADOX</u> A SATELLITE IS IN A CIRCULAR ORBIT ABOUT THE EARTH WHICH ORBIT LIES WITHIN THE UPPER ATMOSPHERE, HENCE IT EXPERIENCES A DRAG FORCE, $\vec{F} = -\alpha \vec{v}$. α IS SMALL, AND THE ORBIT REMAINS ESSENTIALLY CIRCULAR AT ALL TIMES. FIND $v(t)$ AND $r(t)$ IF $r = r_0$ AND $v = v_0$ AT $t = 0$. (r = DISTANCE OF SATELLITE FROM THE CENTER OF THE EARTH)

NOTE: "BRUTE FORCE" USE OF $F = ma$ IS INEFFICIENT HERE.

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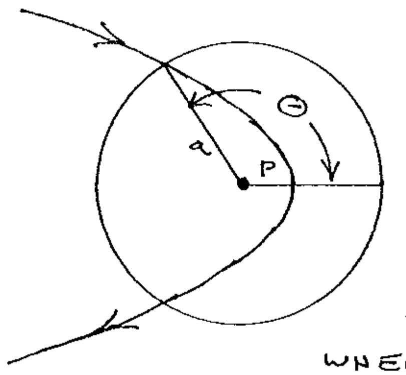
(7) A COMET WITH A <u>PARABOLIC</u> ORBIT HAS PERIHELION DISTANCE p.



SUPPOSE THE EARTH'S ORBIT ABOUT THE SUN HAS RADIUS a.

RECALL THAT A PARABOLA HAS ECCENTRICITY $\epsilon = 1$.

a) WHAT IS THE ANGLE Θ BETWEEN THE LINE JOINING THE SUN AND THE COMET'S PERIHELION, AND THE RADIUS TO THE POINT WHERE THE COMET CROSSES THE EARTH'S ORBIT?

b) WHAT IS THE TOTAL ENERGY OF THE COMET?

c) WHAT IS THE TOTAL ANGULAR MOMENTUM OF THE COMET? (ASSUME THE COMET IS A POINT MASS.)

d) SHOW THAT THE AMOUNT OF TIME THE COMET SPENDS INSIDE THE EARTH'S ORBIT IS $T = \frac{2}{3}\sqrt{2}\sqrt{\frac{a^3}{GM_{sun}}}\sqrt{1-\frac{p}{a}}\left(1+\frac{2p}{a}\right)$

AND THAT THE MAXIMUM POSSIBLE TIME IS $\sim$ 11 WEEKS.

(8) CONSIDER A CENTRAL FORCE $F = -\frac{c}{r^3}$

USE THE ORBIT EQUATION $\frac{d^2u}{d\theta^2} + u = \ldots \qquad u = \frac{1}{r}$

TO SHOW THAT THERE ARE 3 CLASSES OF ORBITS

$$\frac{1}{r} = \frac{1}{r_0}\cos\beta\Theta$$

$$\frac{1}{r} = \frac{1}{r_0} + A\Theta$$

$$\frac{1}{r} = \frac{1}{r_0}e^{\pm\beta\Theta}$$

SKETCH AN ORBIT OF EACH TYPE.

WHAT IS THE RADIUS AND ANGULAR VELOCITY OF THE UNSTABLE CIRCULAR ORBIT AS A FUNCTION OF C?

A FAMOUS PROBLEM IN ASTRONOMY IS THE PRECESSION OF THE PERIHELION OF MERCURY'S ORBIT BY 40" PER CENTURY IN THE DIRECTION OF THE MOTION OF THE ORBIT. THE AVERAGE RADIUS IS 6×10^{10} METERS, THE ECCENTRICITY IS $\epsilon = .206$, AND THE PERIOD IS $T = 0.24$ YEAR.

(9) SUPPOSE THE LAW OF GRAVITY IS REALLY

$$F = -\frac{A}{r^2} - \frac{B}{r^3}$$

USE THE ORBIT EQUATION TO SHOW THAT THE ORBIT HAS THE FORM

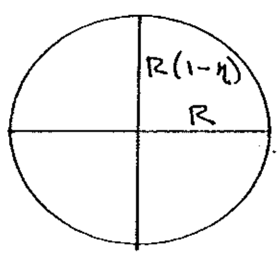
$$\frac{1}{r} = \frac{1 + \epsilon \cos \beta\theta}{a(1-\epsilon^2)}$$

WHERE $\beta = \sqrt{1 - \frac{BM}{L^2}}$

SUPPOSING $\eta \equiv \frac{B}{Aa}$ IS VERY SMALL, SHOW THAT THE PERIHELION ADVANCES BY $\Delta\theta \sim \frac{\pi \eta}{1-\epsilon^2}$ RADIANS EACH REVOLUTION. (INVOKE KEPLER'S 3RD LAW IF NECESSARY).

SHOW THAT $\eta \sim 1.4 \times 10^{-7}$ WOULD BE SUFFICIENT TO EXPLAIN THE PRECESSION OF MERCURY'S ORBIT.

(10) a) <u>OBLATE SUN</u> IF THE SUN HAS EQUATORIAL RADIUS R WHICH IS GREATER THAN THE POLAR RADIUS, $R(1-\eta)$, THEN THE GRAVITATIONAL POTENTIAL BECOMES



$$V = -\frac{GM_1M_2}{r} - \frac{GM_1M_2\,\eta R^2}{5r^3}\left(1 - 3\cos^2\theta\right)$$

IN SPHERICAL COORDS ($\theta = 0$ AT POLE)

CONSIDER ORBITS IN THE PLANE $\theta = 90°$. USE THE EFFECTIVE POTENTIAL METHOD TO SHOW

$$\frac{1}{r_0^3} = \frac{\Omega^2}{GM_1} - \frac{3}{5}\frac{\eta R^2}{r_0^5}$$

AND $\omega = \Omega\sqrt{1 - \frac{6}{5}\frac{GM_1\eta R^2}{\Omega^2 r_0^5}}$ FOR SMALL OSCILLATIONS ABOUT EQUILIBRIUM.

GIVEN $M \sim 2 \times 10^{30}$ kg, $R = 7 \times 10^8$ M, WHAT η IS NEEDED TO EXPLAIN THE ADVANCE OF THE PERIHELION OF MERCURY?

(10) b) GENERAL RELATIVITY EINSTEIN'S THEORY OF GRAVITY AS APPLIED TO PLANETARY MOTION MODIFIES THE NEWTONIAN FORCE LAW TO

$$F = -\frac{G M_1 M_2}{r^2}\left(1 + \frac{3L^2}{(M_2 r c)^2}\right)$$

THE DEPENDENCE OF F ON r IS IDENTICAL TO THAT OF AN OBLATE SUN! IF YOU HAVE WORKED PART a) THEN YOU CAN QUICKLY VERIFY THAT EINSTEIN PREDICTS AN ADVANCE OF $\Delta\theta = \dfrac{24\pi^3 r_0^2}{T^2 c^2}$ PER REVOLUTION, WHICH IS VERY CLOSE TO 40'' PER CENTURY.

R.H. DICKE OF PRINCETON HAS SPENT A LOT OF EFFORT MEASURING THE OBLATENESS OF THE SUN TO SEE IF GENERAL RELATIVITY MIGHT BE WRONG.