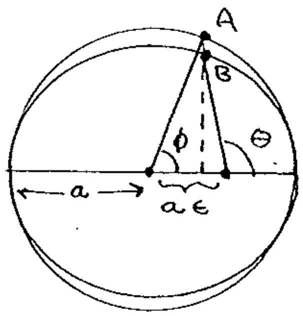
Ph 205 SET 6

DUE: TUES. NOV 6, 1990, MAXIMUM RECORDED SCORE = 70 POINTS
[PROBLEMS (6), (7) AND (10) WILL NOT BE GRADED]

(1) a) KEPLER'S EQUATION OF TIME (SECULAR EQUATION)



AS WELL AS STATING HIS 3 LAWS, KEPLER ALSO GAVE AN EQUATION FOR TIME VS. POSITION IN ELLIPTICAL ORBIT MOTION.

$$K t = \phi - \epsilon \sin \phi$$

K = CONSTANT $\qquad \epsilon$ = ECCENTRICITY.

ϕ = 'ECCENTRIC ANOMALY' = ANGLE FROM CENTER OF ELLIPSE TO POINT A ON A CIRCLE OF RADIUS a SUCH THAT ITS PROJECTION, B, ONTO THE ELLIPSE IS THE DESIRED POSITION.

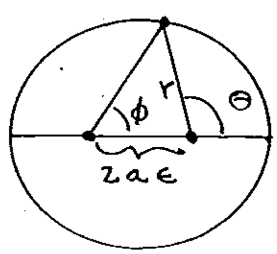
θ = THE USUAL POLAR ANGLE = 'TRUE ANOMALY'.

FOR THE ELLIPSE $\dfrac{1}{r} = \dfrac{1+\epsilon\cos\theta}{a(1-\epsilon^2)}$, FIRST SHOW $\cos\phi = \dfrac{\cos\theta + \epsilon}{1+\epsilon\cos\theta}$

THEN USE $L = Mr^2\dot{\theta}$ TO DERIVE AN EXPRESSION FOR $\dot{\phi}$. THIS SHOULD BE EASILY INTEGRABLE TO YIELD KEPLER'S EQUATION.

SHOW THAT $K = \dfrac{L}{Mab} = \sqrt{\dfrac{GM}{a^3}} = \Omega$ WHERE $\Omega = \frac{2\pi}{T}$ = AVE. ANG. VELOCITY, b = SEMI-MINOR AXIS

b) THE GREEK ECCENTRICITY LET ϕ = ANGLE TO THE PLANET USING THE 'EMPTY' FOCUS AS THE ORIGIN, AS SHOWN. IN THE LIMIT OF SMALL (BUT NON-ZERO) ECCENTRICITY, ϵ, SHOW



$$\dot{\phi} = \frac{L}{Ma^2} = \text{CONSTANT.}$$

THIS RESULT SUPPORTS THE GREEK VIEW THAT THE SUN IS IN UNIFORM MOTION ABOUT A POINT A DISTANCE $2a\epsilon$ FROM THE EARTH. HENCE THE GREEK ECCENTRICITY IS TWICE THE KEPLERIAN.

(2) SPIN-ORBIT COUPLING A PROMINENT ASTRONOMICAL FACT IS THAT THE MOON ALWAYS SHOWS THE SAME FACE TO THE EARTH. THIS MEANS THAT THE MOON ROTATES ONCE ABOUT ITS AXIS EACH MONTH. IT ALSO TURNS OUT THAT THE 'DAYS' OF MERCURY AND VENUS ARE VERY NEARLY EQUAL TO THEIR RESPECTIVE 'YEARS'. IN 1879 G. DARWIN PROPOSED THAT THIS HAS COME ABOUT DUE TO A COUPLING BETWEEN 'DAY' AND 'YEAR' VIA TIDAL FRICTION – RESISTANCE OF THE MOON OR PLANET TO CHANGES IN SHAPE INDUCED BY THE $1/r^2$ VARIATION OF GRAVITY. IN THIS PROBLEM YOU SHOULD DERIVE A KIND OF EXISTENCE PROOF THAT A SPIN-ORBIT COUPLING MECHANISM COULD INDEED PRODUCE THE OBSERVED RESULTS.

FOR SIMPLICITY CONSIDER A POINT SATELLITE OF MASS m WHICH REVOLVES WITH ANGULAR VELOCITY ω AROUND A PLANET OF MASS M. THE PLANET ROTATES ABOUT ITS AXIS WITH ANGULAR VELOCITY Ω. THE TWO AXES ARE PARALLEL.

LET I = MOMENT OF INERTIA OF THE PLANET ABOUT ITS AXIS.
R = DISTANCE FROM THE SATELLITE TO CENTER OF PLANET (CIRCULAR ORBIT).

FIND EXPRESSIONS FOR THE TOTAL ANGULAR MOMENTUM, L, OF THE SYSTEM ABOUT ITS CENTER OF MASS, AND FOR THE TOTAL ENERGY E. THEN ELIMINATE R FROM THESE EXPRESSIONS TO SHOW

$$L = I\Omega + C\,\omega^{-1/3}$$

$$E = \frac{1}{2} I \Omega^2 - \frac{1}{2} C\,\omega^{2/3}$$

WHAT IS THE VALUE OF C?

IN GENERAL THE TWO ANGULAR VELOCITIES ω AND Ω ARE UNEQUAL. SUPPOSE THE MECHANISM OF TIDAL FRICTION REDUCES E IF $\omega \neq \Omega$, BUT CONSERVES ANGULAR MOMENTUM. BY EXAMINING THE BEHAVIOR OF E AS A FUNCTION OF ω, SHOW THAT THERE IS A RANGE OF INITIAL CONDITIONS SO THAT EVENTUALLY $\omega = \Omega$, AND A STABLE FINAL CONFIGURATION OBTAINS.

HINT: YOU MAY WISH TO DEFINE $x = C\omega^{-1/3}$.

FOR THE EARTH-MOON SYSTEM, Ω_e IS DECREASING. GIVE AN EXPRESSION FOR R AS A FUNCTION OF Ω ONLY TO SHOW THAT R INCREASES AS Ω DECREASES. THEN BY KEPLER'S LAW, ω MUST BE DECREASING ALSO, CONSISTENT WITH THE ABOVE ANALYSIS.

DARWIN ALSO SUGGESTED THAT IF THE ABOVE SCENARIO IS EXTRAPOLATED INTO THE PAST, THERE MAY HAVE BEEN A TIME WHEN $R = R_e$ AND THE EARTH AND MOON WERE BOTH PART OF A LARGER PRE-PLANET.... (SEE ALSO SCIENTIFIC AMERICAN, APRIL 1972)
[THIS IDEA HAS BEEN RECENTLY REVIVED, SUPPOSING A 3RD BODY AIDED IN EXTRACTING THE MOON FROM EARTH.]

(3) <u>SKYHOOK</u> AS PART OF THE EQUATORIAL-AFRICAN SPACE PROGRAM, IDI AMIN PROPOSED TO LAUNCH A 'SKYHOOK' SATELLITE. THIS CONSISTS OF A LONG ROPE WHICH IS IN ORBIT SUCH THAT THE ROPE IS ALONG A RADIUS THRU THE CENTER OF THE EARTH, WITH THE LOWER END JUST AT THE SURFACE OF THE EARTH. THE SKYHOOK ORBITS THE EARTH ONCE A DAY – SO IT APPEARS SUSPENDED IN SPACE ABOVE A FIXED POINT ON THE EQUATOR.
"I JUST WANT A PLACE TO HANG MY HAT" – I. AMIN

DERIVE A (DIFFERENTIAL) EQUATION FOR THE TENSION IN THE ROPE. NOTE THAT THE TENSION VANISHES AT THE ENDS TO SHOW

$$r_{MAX} = \frac{r_e}{2}\left(\sqrt{1 + \frac{8GM}{\omega^2 r_e^3}} - 1\right) \quad \text{IF } r_{MIN} = r_e$$

EVALUATE r_{MAX} NUMERICALLY!

IS THE SKYHOOK STABLE AGAINST HANGING A HAT ON IT?

WE SUGGEST 2 WAYS TO ADDRESS THE QUESTION OF STABILITY:

a) CALCULATE KE & PE AND CONSIDER THE VIRIAL THEOREM, OR THE ESCAPE VELOCITY.

b) CONSTRUCT AN EFFECTIVE POTENTIAL. IT IS SUFFICIENT TO RESTRICT YOURSELF TO MOTIONS WHERE THE ROPE AWAYS REMAINS RADIAL - IGNORE COILING...

(4) SOME DIMENSIONAL ANALYSIS:

a) <u>MODELLING</u> WE WISH TO INVESTIGATE SOME PHENOMENON WHICH INVOLVES GRAVITY BY MEANS OF A <u>SCALE MODEL</u>, WHERE ALL LENGTHS ARE SCALED BY $l' = \alpha l$ (USUALLY $\alpha < 1$). HOWEVER WE USE THE SAME MATERIAL TO CONSTRUCT THE MODEL AS USED IN THE ORIGINAL, SO THE DENSITY ρ IS UNCHANGED. IN GENERAL, WE NEED TO STRETCH OR SHRINK THE TIME SCALE, $t' = \beta t$. IN PRACTICE WE DO THIS BY SCALING THE VELOCITIES: $v' = \frac{\alpha}{\beta} v$.

BY WHAT FACTOR MUST WE SCALE THE <u>FORCES</u> SO THAT SIMILAR MOTIONS OCCUR IN THE MODEL & THE ORIGINAL? — SUPPOSING GRAVITY MUST BE INCLUDED AMONG THE FORCES. I.e. WHAT IS β GIVEN α?

A MODEL OF A SHIP WITH $\alpha = 1/N$ IS OBSERVED TO EXPERIENCE A DRAG FORCE F' WHEN TRAVELLING THRU A TANK OF WATER WITH VELOCITY v'. FOR WHAT VELOCITY OF THE ORIGINAL IS THE MODEL RESULT RELEVANT? IF WE SUPPOSE $F = f(v, A)$ WHERE A = AREA OF THE SHIP UNDER WATER, THEN WHAT IS THE FORM OF f SUCH THAT SCALING HOLDS? THAT IS, SUPPOSE $f = K v^p A^q$. WHAT ARE p & q?

b) A PARTICLE STARTS FROM REST SUBJECT TO A FORCE

$$\bar{F} = -A\bar{r} - B\bar{v} \quad , \quad A, B \text{ CONSTANTS.}$$

SHOW BY DIMENSIONAL ANALYSIS THAT THE TIME TO REACH THE ORIGIN IS INDEPENDENT OF THE INITIAL POSITION.

c) A PARTICLE STARTS FROM REST SUBJECT TO A FORCE

$$\vec{F} = -\frac{A\hat{r}}{r^n} \quad . \text{ SHOW THAT } T \sim R^{\frac{n+1}{2}}$$

RELATES THE TIME OF FALL TO THE CENTER FROM INITIAL RADIUS R.

(5) BIO-MECHANICS

a) HOW DOES THE LENGTH OF TIME AN ANIMAL CAN SURVIVE IN A HOT, DRY DESERT DEPEND ON ITS LENGTH (OR HEIGHT) L? EXPLAIN YOUR ANSWER.

b) ANIMALS ARE HEAT ENGINES — THEY MUST SWEAT TO LIVE. HOW DOES 'HORSE POWER' DEPEND ON THE SIZE L OF THE ANIMAL?

c) HOW DOES THE MAXIMUM RUNNING SPEED OF AN ANIMAL DEPEND ON ITS SIZE L IF AIR RESISTANCE IS THE LIMIT? – IF GRAVITY IS THE LIMIT, AS IN RUNNING UPHILL?

d) HOW DOES THE HEIGHT AN ANIMAL CAN JUMP DEPEND ON ITS SIZE L? ASSUME THE RADIUS OF BONES IS PROPORTIONAL TO L.

NOTE: PHYSICS SHOULD NOT CONTRADICT COMMON SENSE. CHECK YOUR ANSWERS AGAINST REALITY.

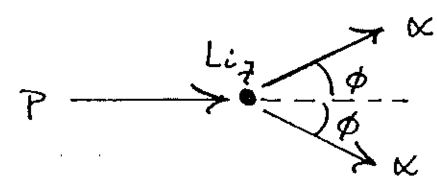
(6) a) PROB 8 p108 B&O.

b) IN A HEAD-ON COLLISION BETWEEN AN ATOM OF MASS M, INITIALLY AT REST, AND AN ELECTRON OF MASS m, INITIAL VELOCITY v_0, WE WISH TO EXCITE THE ATOM TO AN 'ENERGY LEVEL' OF INTERNAL ENERGY E. FIND THE MINIMUM v_0 FOR WHICH THIS IS POSSIBLE, FOR ARBITRARY m/M. YOU SHOULD FIND THAT IF $m \ll M$, INITIAL $KE = E$, BUT IF $m = M$, $KE_i = 2E$....

(7) a) IN A NUCLEAR PHYSICS EXPERIMENT ON THE REACTION

$$P + Li_7 \rightarrow \alpha + \alpha \qquad (\alpha = He_4 \text{ NUCLEUS})$$

$T_P = 0.2$ MeV (MILLION ELECTRON VOLTS), AND THE LITHIUM IS INITIALLY AT REST. INTERNAL ENERGY $Q = 14$ MeV IS LIBERATED IN THIS REACTION. A SPECIAL CASE OF THE FINAL STATE IS WHEN EACH α-PARTICLE MAKES THE SAME ANGLE ϕ TO THE INCIDENT PROTON DIRECTION.



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WHAT IS ϕ, AND V_α = VELOCITY OF THE α'S?

WE MAY AVOID CONVERTING MeV INTO JOULES OR WHATEVER IF WE NOTE THAT $E = mc^2$ AND THAT $M_p = 938\ \text{MeV}/c^2$

$M_\alpha = 3728\ \text{MeV}/c^2$

$M_{Li} = 6539\ \text{MeV}/c^2$

THEN, FOR EXAMPLE, $T = \frac{1}{2}mv^2 = \frac{1}{2}mc^2\left(\frac{v}{c}\right)^2$

SO USING THESE FUNNY UNITS THINGS WILL BE SIMPLE IF WE ALWAYS MEASURE VELOCITIES RELATIVE TO c = SPEED OF LIGHT.

ANS: $\cos\phi = \sqrt{\dfrac{M_p T_p}{2M_\alpha (T_p + Q)}}$

b) PROB 16, p109 B&O

(8) a) PROB 17, p109 B&O. WHAT IS $\dfrac{d\sigma}{d\cos\theta}$, $\dfrac{d\sigma}{d\Omega}$?

NOTE THAT YOUR RESULT REALLY HOLDS IN THE CENTER OF MASS FRAME, RATHER THAN THE LAB FRAME.

b) USE PART a) TO SHOW THAT THE AVERAGE FORCE ON A SPHERE DUE TO ELASTIC COLLISIONS WITH AIR MOLECULES IS THE SAME AS FOR INELASTIC COLLISIONS, SUPPOSING $M_{MOL} \ll M_{SPHERE}$. HENCE THE RESULT OF PROB (7), SET 1 HOLDS FOR ELASTIC AS WELL AS INELASTIC COLLISIONS.

c) IN THE LAB FRAME, WHAT IS $d\sigma/dT_{1\,LOST}$ = CROSS SECTION OF BB'S STRIKING THE SPHERE AS A FUNCTION OF KINETIC ENERGY LOST BY THE BB'S?

HINT: KE LOST BY BB'S = KE GAINED BY SPHERE.

ANS: $\dfrac{d\sigma}{dT} = \dfrac{\pi R^2}{T_{MAX}}$. WHAT IS T_{MIN}, T_{MAX}?

(9) a) PROB 18, p153 B&O. THERE IS NO NEED TO INTEGRATE OVER RUTHERFORD'S FORMULA – GO BACK TO THE BASICS! WHAT WOULD THE CROSS SECTION BE FOR A REPULSIVE FORCE $F = +GMm/r^2$?

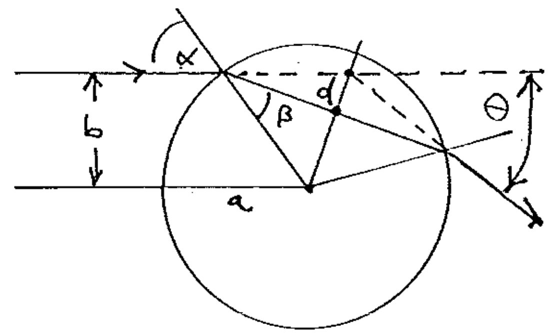
b) PROB 19, p153 B&O. IN THE JARGON, YOU ARE CALCULATING THE 'CAPTURE CROSS-SECTION'!

(10) CALCULATE THE DIFFERENTIAL CROSS SECTION FOR SCATTERING OFF THE POTENTIAL

$$V(r) = \begin{cases} 0 & r > a \\ -V_0 & r < a \end{cases}$$

THE TRICK IS TO USE THE PROBLEM ON p16 OF L&L TO NOTE THAT THIS IS EQUIVALENT TO LIGHT SCATTERING OFF A SPHERE OF INDEX OF REFRACTION $n = \sqrt{\frac{T}{T+V}}$ ($n = 1$ OUTSIDE THE SPHERE, $n > 1$ INSIDE)

TO RELATE IMPACT PARAMETER b TO SCATTERING ANGLE Θ, YOU MIGHT SOLVE FOR DISTANCE d, SHOWN IN THE FIGURE, IN 2 WAYS



$$\Rightarrow b^2 = \frac{a^2 n^2 \sin^2 \Theta/2}{n^2 + 1 - 2n \cos \Theta/2}$$

WHAT IS Θ_{MAX}?

THEN FOR $n > 1$, YOU SHOULD FIND

$$\frac{d\sigma}{d\cos\Theta} = \frac{\pi a^2}{2} \frac{n^2}{\cos \Theta/2} \frac{(n \cos \Theta/2 - 1)(n - \cos \Theta/2)}{(n^2 + 1 - 2n \cos \Theta/2)^2}$$

SKETCH THIS FOR $n = 2$, NOTING Θ_{MAX} TO SHOW $\sigma_{TOT} = \pi a^2$.

THE CASE $n < 1$ IS NOT POSSIBLE FOR LIGHT SCATTERING IN AIR, BUT WE COULD ARRANGE $n_{OUTSIDE} > n_{INSIDE} \geq 1$ BY SCATTERING OFF AN AIR BUBBLE UNDER WATER. NOW THERE IS NO LIMIT TO THE SCATTERING ANGLE 'AT FIRST GLANCE'. HOWEVER, WHEN b IS LARGE, THE LIGHT WON'T ENTER THE BUBBLE, BUT JUST BOUNCES OFF THE SURFACE BY 'TOTAL EXTERNAL REFLECTION.' TRY LOOKING THRU A BUBBLE IN A SWIMMING POOL SOMETIME. THE REFLECTION SCATTERING IS ISOTROPIC (OVER THE ALLOWED RANGE OF ANGLES) AS IN PROB (8).

IF ENERGETIC, SKETCH THE RESULT FOR THE CASE $n = 1/2$ INCLUDING BOTH REFLECTION AND REFRACTION.

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(11) CALCULATE THE DIFFERENTIAL CROSS SECTION FOR SCATTERING OFF THE REPULSIVE CENTRAL FORCE WHOSE POTENTIAL IS $V = \dfrac{\alpha}{r^2}$. THE BEAM PARTICLES HAVE ENERGY E WHEN r IS LARGE.

THIS PROBLEM IS WELL SUITED TO THE FORMAL METHOD DISCUSSED IN SEC 18 OF L&L.

YOU SHOULD FIND THAT THE IMPACT PARAMETER b OBEYS

$$b^2 = \frac{\alpha}{E}\,\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)} \qquad \Theta = \text{SCATTERING ANGLE}$$

AND HENCE

$$\frac{d\sigma}{d\Omega} = \frac{\pi^2\alpha}{E\sin\Theta}\,\frac{\pi-\Theta}{\Theta^2(2\pi-\Theta)^2}$$

SKETCH THIS. AS FOR RUTHERFORD SCATTERING, SINCE F EXTENDS TO $r = \infty$, THE TOTAL CROSS SECTION DIVERGES.

B & O PROBLEMS:

p108

8. A beam of hydrogen molecules moves along the $\hat{z}$ direction with a kinetic energy of 1 electron volt (eV) per molecule. The molecules are in an excited state, from which they can decay and dissociate into two hydrogen atoms. When the velocity of a dissociation atom is perpendicular to the $\hat{z}$ direction, its energy is always 0.8 eV. Calculate the energy per molecule released in the dissociative reaction.

p109

16. A neutron with kinetic energy T_0 collides with a proton at rest. If the differential cross section for elastic scattering in the laboratory system is $d\sigma/d\theta = A \sin 2\theta$, what is the distribution $d\sigma/dT$, where T is the recoil kinetic energy of the proton? Assume equal mass for the proton and neutron. Note that $\theta \le \pi/2$ in this case.

17. Calculate the differential cross section $d\sigma/d\theta$ for the scattering of BBs from a steel sphere of radius R. Ignore the effects of gravity.

p153

18. A parallel beam of small projectiles is fired from space toward the moon with initial velocity v_0. What is the collision cross section σ for the projectiles to hit the moon? Express σ in terms of the moon's radius R_L, the escape velocity from the moon v^L_{esc}, and v_0. Neglect the motion of the moon.

19. The interaction between an atom and an ion at distances greater than contact is given by the potential energy $V(r) = -C/r^4$. [$C = (e^2/2)P_a{}^2$, where e is the ion charge and P_a is the polarizability of the atom.] Make a sketch of the effective potential energy vs. the radial coordinate. Note that if the total energy of the ion exceeds the maximum value of the effective potential energy, the ion spirals inward to the atom. Find the cross section for an ion of velocity v_0 to strike an atom. Assume that the ion is much lighter than the atom.