1. The Lagrangian for this system is given by (except for damping terms)

$$L = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_1 x_2^2 - \frac{1}{2} k(x_2 - x_1)^2 + x_1 F_e \cos\omega t$$

The Euler-Lagrange equation gives two coupled equations, (We add damping terms here)

$$m\ddot{x}_1 + b\dot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_e \cos\omega t$$
$$m\ddot{x}_2 + b\dot{x}_2 + k_1 x_2 + k_2(x_2 - x_1) = 0$$

We define $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$. We add the above two equations and subtract them. Then,

$$\ddot{q}_1 + 2\beta\dot{q}_1 + \omega_1^2 q_1 = \frac{F_e}{m}\cos\omega t$$
$$\ddot{q}_2 + 2\beta\dot{q}_2 + \omega_2^2 q_2 = \frac{F_e}{m}\cos\omega t$$

where we used notations $\omega_1^2 = k_1/m$ and $\omega_2^2 = \frac{k_1 + 2k_2}{m}$. Assuming that we retain only real part of the solution we can write $q_1 = Ae^{i\omega t}$ and $q_2 = Be^{i\omega t}$. We put this expression into the above equations and get,

$$-\omega^2 q_1 + 2i\omega\beta q_1 + \omega_1^2 q_1 = \frac{F_e}{m} e^{i\omega t} \Rightarrow A = \frac{F_e/m}{\omega_1^2 - \omega^2 + 2i\beta\omega}$$
$$-\omega^2 q_2 + 2i\omega\beta q_2 + \omega_2^2 q_2 = \frac{F_e}{m} e^{i\omega t} \Rightarrow B = \frac{F_e/m}{\omega_2^2 - \omega^2 + 2i\beta\omega}$$

Thus,

$$x_1 = \frac{1}{2}(A+B)e^{i\omega t} = \frac{F_e}{m}\,\frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{\omega_1^2 - \omega^2 + 2i\beta\omega}\,\frac{e^{i\omega t}}{\omega_2^2 - \omega^2 + 2i\beta\omega}$$
$$x_2 = \frac{1}{2}(A-B)e^{i\omega t} = \frac{F_e}{2m}\,\frac{\omega_2^2 - \omega_1^2}{\omega_1^2 - \omega^2 + 2i\beta\omega}\,\frac{e^{i\omega t}}{\omega_2^2 - \omega^2 + 2i\beta\omega}$$

where we used $x_1 = \frac{1}{2}(q_1 + q_2)$, $x_2 = \frac{1}{2}(q_1 - q_2)$, $\omega_0^2 = (k_1 + k_2)/m$

2. Let $\vec{F}_1$ and $\vec{F}_2$ denote the force acted by the left spring and by the right spring, respectively. Then, the condition of the equilibrium is,

$$\vec{F}_1 + \vec{F}_2 = 0$$

Thus, the only possible conditions for equilibrium is either i)

$$\cancel{\vec{F}_1 = \vec{F}_2 = 0} \quad \vec{F}_1 = -\vec{F}_2 \neq 0$$

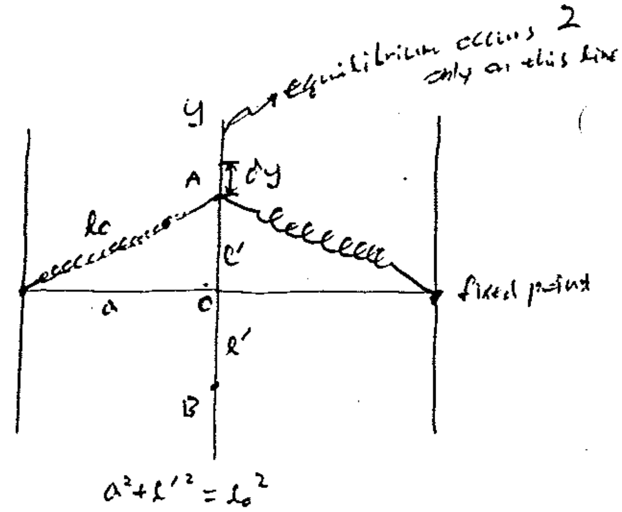
or, ii)

$$\vec{F}_1 = \vec{F}_2 = 0$$

Thus, we immediately see that the particle should be on the line which is perpendicular to the line connecting two fixed points and passes through the central point since the force is generated by springs.

(a) In this case, there is one equilibrium which satisfies (i), i.e., the point O in the figure shown in next page. Additionally, there are two more equilibrium points which correspond to the condition (ii) as shown in the figure. Consequently, there are three equilibrium positions. As to O, since each spring pushes the mass m, the deviatio from the equilibrium point results the pushing of mass m away from O. Thus, it is unstable equilibrium. As to A and B, the Lagrangian can be written as,

$$L = \frac{1}{2} m \dot{\delta y}^2 + \frac{(2k)}{2} \cdot \left(\sqrt{a^2 + (\pm l' + \delta y)^2} - l_0\right)^2$$



Using the fact that δy is small, we can expand,

$$\left(\sqrt{(a^2 + l'^2 \pm 2l'\delta y)} - l_0\right)^2 \approx \left(\frac{l'}{l_0}\delta y\right)^2$$

$$\left(\text{cf. } a^2 + l'^2 = l_0^2,\ \sqrt{1+x} = 1 + \frac{x}{2} + \cdots\right)$$

Thus, Lagrangian can be written as,

$$L = \frac{1}{2} m \dot{\delta y}^2 + \frac{(2k)}{2} \frac{l_0^2 - a^2}{l_0^2} (\delta y)^2$$

which gives the small oscillation frequency,

$$\omega^2 = \frac{2k}{m}\left(1 - \frac{a^2}{l_0^2}\right)$$

We can be sure that This is a stable equilibrium since the sign of the right hand side is positive.

(b) In this case, we can immediately see the second condition (ii) can not be satisfied, since a is larger than the equilibrium length. Next y = 0, the Lagrangian can be written as,

$$L = \frac{1}{2} m \dot{\delta y}^2 + \frac{(2k)}{2}\left(\sqrt{a^2 + (\delta y)^2} - l_0\right)^2$$

We approximate,

$$\sqrt{a^2 + (\delta y)^2} = a\sqrt{1 + (\delta y)^2/a^2}$$
$$\approx a\left(1 + \frac{(\delta y)^2}{2a^2}\right)$$

Thus, the Lagrangian can be rewritten as,

$$L = \frac{1}{2} m \dot{\delta y}^2 + \frac{(2k)}{2}\left((a - l_0)^2 + \frac{a - l_0}{a}(\delta y)^2\right)$$

which gives stable oscillation frequency,

$$\omega^2 = \frac{2k}{m}\left(1 - \frac{l_0}{a}\right)$$

Thus, there is one stable equilibrium and only one equilibrium.

(c) If $a = l_0$, Lagrangian can be written as,

$$L = \frac{1}{2} m \dot{y}^2 + \frac{2k}{2}\left(\sqrt{l_0^2 + y^2} - l_0\right)^2$$

Assuming that y is much less than l_0 (small oscillation), we can expand,

$$\left(\sqrt{l_0^2 + y^2} - l_0\right)^2 \approx \left(\frac{y^2}{2l_0}\right)^2$$
$$= \frac{y^4}{4l_0^2}$$

Thus, L can approximately be written as,

$$L = \frac{1}{2} m \dot{y}^2 + \frac{k y^4}{4 l_0^2}$$

The parameters in this problem are amplitude A and $\frac{k}{m l_0^2}$ which appears in the equation of motion,

$$\ddot{y} + \frac{k}{m l_0^2} y^3 = 0$$

Sincer there is no dimensionless parameters possible in this problem except pure numbers, we can simply set,

$$T = A^\alpha \left(\frac{k}{m l^2}\right)^\beta$$

The dimensional analysis of each sides shows that

$$2\beta = -1, \quad \alpha - 2\beta = 0. \quad \Rightarrow \alpha = -1, \ \beta = -\tfrac{1}{2}.$$

Thus, the period is inversely proportional to the amplitude.

$$T \propto \frac{1}{A}.$$

Now we consider the succesive approximation. First thing we should do is to find proper form of approximation. From the additional dimensional analysis, we find the form of the most general solution of this equation is,

$$y(t) = a\, f\left(\omega_0 \frac{a}{l} t\right)$$

where $\omega_0^2 = k/m$. Since the solution is periodic, we can Fourier-expand the solution in the form of (a_n, b_1, c_n; pure numbers)

$$y(t) = a \sum_{n=0}^{\infty} a_n \cos\left(b_1 n \omega_0 \frac{a}{l_0} t\right) + a \sum_{n=1}^{\infty} c_n \sin\left(b_1 n \omega_0 \frac{a}{l_0} t\right)$$

From $y \to -y$ symmetry of the governing equation, we find that $c_n = 0$. Additionally, by the same method, $a_{\text{odd number}} \neq 0$, and otherwise $a_n = 0$. Here, by "successive approximation", we mean we neglect all a_n with n>1 and concern ourself only with the leading order term in Fourier expansion. Without loss of generality we can set $a_1 = 1$. We put this y into the equation of motion and get,

drop higher order harmonic

$$\ddot{y} + \frac{k}{m l_0^2} y^3 = 0 \Rightarrow -b_1^2 \omega_0^2 \frac{a^2}{l_0^2} \cos\left(\omega_0 b_1 \frac{a}{l_0} t\right) + a^3 \frac{\omega_0^2}{l_0^2} \cos^3\left(\omega_0 b_1 \frac{a}{l_0} t\right) \approx \left(\frac{3}{4} - b_1^2\right) a^3 \frac{\omega^2}{l_0^2}$$

where we used $\cos\left(\omega_0 b_1 \frac{a}{l} t\right) = 0$. Thus, the period satisfies,

$$\frac{3}{4} - b_1^2 = 0, \Rightarrow \quad b_1 \omega_0 \frac{a}{l_0} T = 2\pi = \frac{\sqrt{3}}{2} \sqrt{\frac{k}{m}} \frac{a}{l_0} T = 2\pi$$

$$\therefore T = 4\pi \frac{\sqrt{3}}{3} \frac{l_0}{a} \sqrt{\frac{m}{k}}.$$

comment. This problem is a little bit confusing since the meaning of "succssive approximation"is not so clear from the context of the problem.

3. (a) We consider the half period motion during which the particle is moving from x = − a to x = a'. Since the speed is positive in this case, the governing equation becomes,

$$m\ddot{x} = -kx - \mu m g \quad , \quad \omega_0^2 \equiv \frac{k}{m}$$

The above solution is easily solved to yield,

$$x = A\cos\omega_0 t + B\sin\omega_0 t - \frac{\mu g}{\omega_0^2}$$

Using the initial conditions,

$$x(0) = a \;, \quad \dot{x}(0) = 0.$$

the full solution is determined to give,

$$x = \left(\frac{\mu g}{\omega_0^2} - a\right)\cos\omega_0 t - \frac{\mu g}{\omega_0^2}$$

Thus, the loss of amplitude is,

$$\Delta a = |a' - a| = \left|\left(a - 2\frac{\mu g}{\omega_0^2}\right) - a\right| = \frac{2\mu g}{\omega_0^2}$$

Consequently, the total time required to reach a complete amplitude drop is,

$$\frac{T}{2}\cdot\frac{a}{\Delta a} = \frac{T\cdot a}{4\mu g}\omega_0^2 = \frac{\pi}{2}\frac{A_0\omega_0}{\mu g}$$

(b) In general cases, the equation of motion can be written as,

$$\ddot{x} + \omega_0^2 x = -\mu g\, \mathrm{sgn}(\dot{x})$$

where sgn(x) = 1 for x>0 and −1 for x<0. Thus, the damping function is,

$$\epsilon f = -\mu g\, \mathrm{sgn}(\dot{x})$$

Using the formula derived in the notes, if we write x = a(t) cos ϕ (t), $(\dot{x} = -a\omega_0\sin\phi)$

$$\dot{a} = -\frac{1}{2\pi\omega_0}\int_0^{2\pi}\epsilon f\sin\phi\, d\phi = \frac{\mu g}{2\pi\omega_0}\int_0^{2\pi}\mathrm{sgn}(-\sin\phi)\sin\phi\, d\phi$$

$$= \frac{\mu g}{2\pi\omega_0}\left[\int_0^{\pi} -\sin\phi\, d\phi + \int_{\pi}^{2\pi}\sin\phi\, d\phi\right] = -\frac{2\mu g}{\pi\omega_0}$$

and

$$\dot{\varphi} = \omega_0 - \frac{\epsilon}{2\pi\omega_0 a}\int_0^{2\pi} f\cos\phi\, d\phi = \omega_0 + \frac{\mu g}{2\pi\omega_0 a}\left[\int_0^{\pi} -\cos\phi\, d\phi + \int_{\pi}^{2\pi}\cos\phi\, d\phi\right]$$

$$= \omega_0$$

Using the initial conditions a(0) = A_0 and φ(0) = 0, a(t) and φ(t) are determined as,

$$a(t) = A_0 - \frac{2\mu g}{\pi\omega_0}t \qquad \varphi(t) = \omega_0 t$$

Thus, the solution we would like to find is,

$$x = \left(A_0 - \frac{2\mu g}{\pi\omega_0}t\right)\cos\omega_0 t$$

4. (a) We consider the half period motion during which the speed of the particle is negative. Then, the equation of motion is,

$$\ddot{x} + \omega_0^2 x = \beta\dot{x}^2$$

The zeroth order equation is,

$$\ddot{x}^{(0)} + \omega_0^2 x^{(0)} = 0$$

which gives a solution

$$x^{(0)} = A_0\cos\omega_0 t$$

which satisfies the boundary condition x = A_0 and $\dot{x}$ = 0 at t = 0. Writing $x = x^{(0)} + \beta\ x^{(1)}$, we put this expression into equation of motion and retain only linear order terms. Then,

$$x^{(1)} + \omega_0^2 x^{(1)} = \omega_0^2 A^2 \sin^2 \omega_0 t = \frac{\omega_0^2 A^2}{2} - \frac{\omega_0^2 A^2}{2} \cos 2\omega_0 t$$

We can guess the form of the solution is $x = \frac{A^2}{2} + B \cos 2\omega_0 t$. We put this into the above equation to get,

$$(-4\omega_0^2 + \omega_0^2) B \cos 2\omega_0 t = -\frac{\omega_0^2 A^2}{2} \cos 2\omega_0 t \Rightarrow B = \frac{A^2}{6}.$$

which gives,

$$x^{(1)} = \frac{A^2}{2} + \frac{A^2}{6} \cos 2\omega_0 t$$

Thus, up to the first order in β, the solution can be written as,

$$x = A \cos \omega_0 t + \beta \left(\frac{A^2}{2} + \frac{A^2}{6} \cos 2\omega_0 t \right)$$

At t=0, the initial amplitude is,

$$A + \beta \cdot \frac{2}{3} A^2 = A_0$$

If β =0, we get A = A_0. Thus, we write $A = A_0 + A_1 \beta$. We put this expression into the above equation and retain only upto leading order in β. Then,

$$A_1 + \frac{2}{3} A_0^2 = 0$$

which gives the value of $A_1 = -\frac{2}{3} A_0^2$. Thus, $A = A_0 (1 - \frac{2}{3} \beta A_0)$. At $\omega_0 t = \pi$ when speed of the mass is 0 again, the amplitude is,

$$|x| = \left| -(A_0 - \tfrac{2}{3}\beta A_0) + \beta \left(\frac{A^2}{2} + \frac{A^2}{6} \right) \right| = \left| -(A_0 - \tfrac{4}{3} \beta A_0^2) \right| = A_0 (1 - \tfrac{4}{3} \beta A_0)$$

(b) The damping function in this case is given by

$$\epsilon f = -\beta \dot{x} |\dot{x}|$$

Following the conventional method, we write $x = a(t) \cos \phi(t)$ and $\dot{x}(t) = -\omega_0 a(t) \sin \phi$.

Thus,

$$\dot{a} = -\frac{1}{2\pi\omega_0} \int_0^{2\pi} \epsilon f \sin\phi\, d\phi = \frac{\beta a^2 \omega_0^2}{2\pi\omega_0} \int_0^{2\pi} \mathrm{sgn}(-\omega_0 a(t) \sin\phi) \sin\phi\, d\phi$$

$$= \frac{\beta a^2 \omega_0}{2\pi} \left(-\int_0^{\pi} \sin^3\phi\, d\phi + \int_{\pi}^{2\pi} \sin^3\phi\, d\phi \right) = -\frac{\beta a^2 \omega_0}{\pi} \int_0^{\pi} (1 - \cos^2\phi) \sin\phi\, d\phi$$

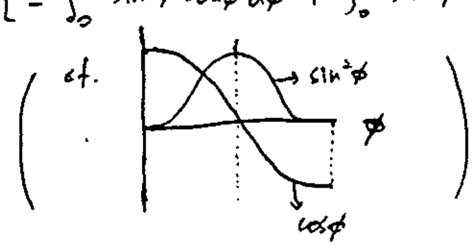
$$= -\frac{4}{3\pi} \beta \omega_0 a^2 \qquad (1)$$

and

$$\dot{\phi} = \omega_0 - \frac{1}{2\pi\omega_0 a} \int_0^{2\pi} \epsilon f \cos\phi\, d\phi = \omega_0 + \frac{\beta a^2 \omega_0^2}{2\pi a \omega_0} \left[-\int_0^{\pi} \sin^2\phi \cos\phi\, d\phi + \int_0^{\pi} \sin^2\phi \cos\phi\, d\phi \right]$$

$$= \omega_0$$

$$\therefore \phi = \omega_0 t$$



From (1), we have

$$-\frac{da}{a^2} = \frac{4}{3\pi} \beta \omega_0$$

which can be integrated to yield,

$$\frac{1}{a(t)} = \frac{1}{A} + \frac{4}{3\pi} \beta \omega_0 t$$

where we used the initial condition a(0) = A_0. Thus, the full solution is,

$$x = a(t) \cos \omega_0 t$$

5. We are given the equation of motion,

$$\frac{d^2u}{d\theta^2} + u = \frac{m\alpha}{L^2} + f \qquad (1)$$

Now we set,

$$u = \frac{m\alpha}{L^2}\left(1+\epsilon(\theta)\cos\phi(\theta)\right)$$

By the direct differentiation with respect to θ , we get

$$\dot{u} = \frac{m\alpha}{L^2}\left(\dot{\epsilon}\cos\phi - \epsilon\sin\phi\,\dot{\phi}\right) = -\frac{m\alpha}{L^2}\epsilon\sin\phi \qquad (2)$$

where we equate the derivative with the given form in the problem. We take the differentiation again and get

$$\ddot{u} = -\frac{m\alpha}{L^2}\left(\dot{\epsilon}\sin\phi + \epsilon\cos\phi\,\dot{\phi}\right)$$

We put this into (1) and get,

$$\dot{\epsilon}\sin\phi + \epsilon\cos\phi\,\dot{\phi} - \epsilon\cos\phi = -\frac{L^2}{m\alpha^2}f$$

$$\dot{\epsilon}\cos\phi - \epsilon\sin\phi\,\dot{\phi} + \epsilon\sin\phi = 0$$

where the second equation is obtained from(2). Sinϕ (3) + cosϕ (4) gives,

$$\dot{\epsilon} = -\frac{L^2}{m\alpha^2} f \sin\phi$$

and cos ϕ (3) - sinϕ (4) gives,

$$\dot{\phi} = 1 - \frac{L^2}{m\alpha^2\epsilon} f\cos\phi$$

Assuming that f is small, we take one period average. Notice also that although ϵ has θ dependence we can pull that factor out of the integral since it is assumed to be slow varying. Thus, we get,

$$\dot{\epsilon} = -\frac{L^2}{2\pi m\alpha}\int_0^{2\pi} f\sin\phi\, d\phi$$

$$\dot{\phi} = 1 - \frac{L^2}{2\pi m\alpha\epsilon}\int_0^{2\pi} f\cos\phi\, d\phi$$

Now, we set,

$$f = \frac{\beta m u^2}{L^2} = \frac{\beta m}{L^2}\left(\frac{m\alpha}{L^2}(1+\epsilon\cos\theta)\right)^2$$

Then,

$$\dot{\phi} = 1 - \frac{\alpha\beta m^2}{2\pi\epsilon L^4}\int_0^{2\pi}(1+\epsilon\cos\phi)^2\cos\phi\, d\phi = 1 - \frac{\alpha\beta m^2}{2\pi\epsilon L^4}\int_0^{2\pi}(\overset{0}{1} + \underset{2\pi\epsilon\ (\text{from } \cos^2\phi+\sin^2\phi=1)}{2\epsilon\cos\phi} + \overset{0}{\cos^2\phi})\cos\phi\, d\phi$$

$$= 1 - \frac{\alpha\beta m^2}{L^4} \quad \Rightarrow \quad \phi = \left(1-\frac{\alpha\beta m^2}{L^4}\right)\theta$$

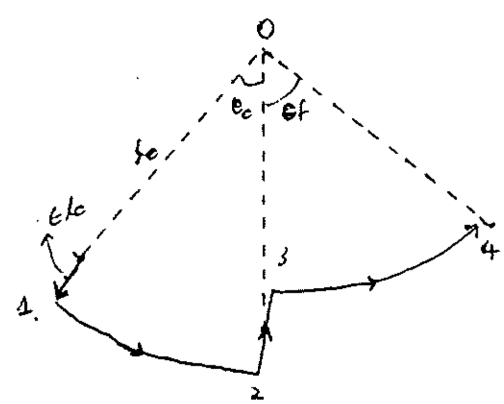
$$\dot{\epsilon} = -\frac{\alpha\beta m^2}{2\pi L^4}\int_0^{2\pi}(1+\epsilon\cos\phi)^2\sin\phi\, d\phi = 0 \quad \Rightarrow \quad \epsilon = \epsilon$$

Thus, the orbit equation is given by,

$$u = \frac{m\alpha}{L^2}\left(1 + \epsilon\cos\left(1-\frac{\alpha\beta m^2}{L^4}\right)\theta\right)$$

6. The Swing

(a) Referring to the right figure, at point 1, the total energy is given by



$$E = -mg(1+\epsilon)l_c\cos\theta_c$$

While the mass moves from 1 to 2, the total energy is conserved. Thus, the angular speed of the particle at point 2 can be calculated via the energy conservation. That is,

$$-mg(1+\epsilon)l_0\cos\theta_0 = -mg(1+\epsilon)l_0 + \frac{1}{2}ml_0^2(1+\epsilon)^2\dot{\phi}_2^2 \Rightarrow \dot{\phi}_2^2 = \frac{2g}{l_0(1+\epsilon)}(1-\cos\theta_0)$$

If the mass is pulled from O near the angle θ is 0 abruptly, the force acted on the mass is radial force. Thus, during this process, the angular momentum should be conserved. That gives,

$$L^2 = m^2l_0^4(1+\epsilon)^4\dot{\phi}_2^2 = m^2l_0^4(1-\epsilon)^4\dot{\phi}_3^2 \Rightarrow \dot{\phi}_3^2 = \frac{(1+\epsilon)^4}{(1-\epsilon)^4}\dot{\phi}_2^2$$

While the mass moves from 3 to 4, the total energy is conserved/ Thus, energy conservation gives,

$$-mg(1-\epsilon)l_0 + \frac{1}{2}ml_0^2(1-\epsilon)^2\dot{\phi}_3^2 = -mg(1-\epsilon)l_0\cos\theta_f \Rightarrow \dot{\phi}_3^2 = \frac{2g}{l_0(1-\epsilon)}(1-\cos\theta_f)$$

Assuming that the motion is small oscillation, we get

$$(1-\cos\theta_f) = \left(\frac{1+\epsilon}{1-\epsilon}\right)^3(1-\cos\theta_0) \Rightarrow \theta_f^2 = \left(\frac{1+\epsilon}{1-\epsilon}\right)^3\theta_0^2$$

Since this pumping has been achieved during the half period, we can say that

$$\Delta\theta = \theta_f - \theta_0 = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{3/2}\theta_0 - \theta_0 = (1+3\epsilon)\theta_0 - \theta_0 = 3\epsilon\theta_0$$

during

$$\Delta t = \frac{\pi}{\omega_0}$$

Thus, we have,

$$\frac{d\theta}{dt} \approx \frac{\Delta\theta}{\Delta t} = \frac{3\epsilon\omega_0}{\pi}\theta \Rightarrow \theta = \theta_0\exp\left(\frac{3\epsilon\omega_0}{\pi}t\right) \quad \left(\omega_0 = \sqrt{\frac{l}{g}}\right)$$

(b) The Lagrangian of this system can be written as,

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\theta^2$$

for small oscillation where $l = l_0(1+\epsilon\sin 2\omega_0 t)$ The Euler-Lagrange equation gives,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial}{\partial\theta}L = 0 \Rightarrow \frac{d}{dt}(l^2\dot{\theta}) + gl\theta = 0$$

Putting the expression for l into the above equation and retaining only terms of up to the linear order in ϵ , we get

$$\ddot{\theta} + \omega_0^2\theta = \epsilon(\omega_0^2\theta\sin 2\omega_0 t - 4\omega_0\dot{\theta}\cos 2\omega_0 t)$$

where $\omega_0^2 = l/g$. Now we use the method of average. We assume the solution of the type,

$$\theta = a(t)\cos\phi(t) = a(t)\cos\omega_0 t, \quad \dot{\theta} = -\omega_0 a(t)\sin\omega_0 t$$

The damping function can be written as,

$$f = \omega_0^2\theta\sin 2\omega_0 t - 4\omega_0\dot{\theta}\cos 2\omega_0 t \approx \omega_0^2\cdot a(\cos\phi\sin 2\phi + 4\sin\phi\cos 2\phi)$$

up to the linear order in ϵ . Adopting the formula from the note, we have,

$$\dot{a} = -\frac{\epsilon\cdot a\omega_0^2}{2\pi\omega_0}\int_0^{2\pi}(\cos\phi\sin 2\phi + 4\sin\phi\cos 2\phi)\sin\phi\, d\phi$$

$$= -\frac{\epsilon\cdot a\omega_0^2}{2\pi\omega_0}\int_0^{2\pi}\left(\frac{\sin^2 2\phi}{2} + 4\sin^2\phi - 8\sin^4\phi\right)d\phi$$

$$= -\frac{\epsilon\cdot a\omega_0^2}{2\pi\omega_0}\left(\frac{\pi}{2} + 4\pi - 8\,\frac{3\cdot 1}{4\cdot 2}\,2\pi\right) = \frac{3}{4}\epsilon\omega_0 a$$

where we used a formula found in integral tables,

$$\int_0^{2\pi} \sin^{2m}\theta \, d\theta = \frac{(2m-1)!!}{(2m)!!} 2\pi$$

Above equation can easily be solved to give,

$$a(t) = a_0 \exp\left(\frac{3}{4}\epsilon\omega_0 t\right)$$

Thus,

$$\theta = \theta_0 \exp\left(\frac{3}{4}\epsilon\omega_0 t\right)\cos\omega_0 t.$$

7. (a) In the accelerated frame with origin at x_1, the equation of motion is,

$$m\ddot{x}_2 = -k(x_2 - l_0) - m\ddot{x}_1$$

where the second term on the right hand side represents the fictitious force due to the acceleration of the origin. Using $x_1 = a\cos\omega t$, we can rewrite above equation in the form, $(\omega_0^2 = k/m)$

$$\ddot{x}_2 + \omega_0^2 x_2 = \omega_0^2 l + a\omega^2\cos\omega t$$

We can guess the form of the solution as,

$$x_2 = l + \alpha\cos\omega t$$

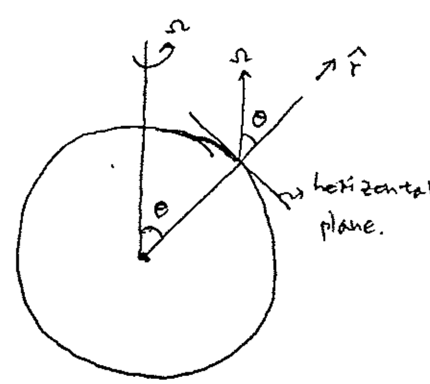
and put this into the original equation to get,

$$\alpha(\omega_0^2 - \omega^2) = a\omega^2$$

$$\therefore x_2 = l + \frac{a\omega^2}{\omega_0^2 - \omega^2}\cos\omega t$$

Thus, the full solution is,

$$x = x_1 + x_2 = l + \frac{a\omega_0^2}{\omega_0^2 - \omega^2}\cos\omega t$$



(b) We ignore the centrifugal force. Then, the equation of motion is (due to Coriolis force)

$$m\frac{d^2}{dt^2}\vec{r} = -2m\vec{\Omega}\times\vec{v} + \vec{N} \text{ (normal force from the plane)}$$

Clearly, since the particle moves on a smooth horizontal plane, only the projection of angular momentum along the $\hat{r}$ direction can contribute. Thus, the above equation can be written as,

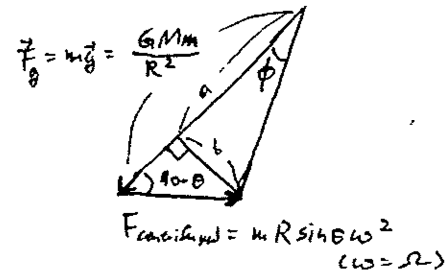
$$m\frac{d}{dt}\vec{v} = -2m\cos\theta\,\Omega\,\hat{r}\times\vec{v}$$

Thus the angular frequency of the motion is $2\Omega\cos\theta$ and the particle moves in a circle. (See p. 170 of notes) Thus, the radius of the circle is,

$$r = \frac{v}{\text{angular frequency}} = \frac{v}{2\Omega\cos\theta}$$

8. (a) Since the plumb is not moving, the forces concerned are pure gravity and the centrifugal force which are depicted right. Referring to the right figure, we find that

$$\tan\phi = \frac{b}{a} = \frac{mR\sin\theta\,\omega^2\cdot\sin(90-\theta)}{\frac{GMm}{R^2} - mR\sin\theta\,\omega^2\cos(90-\theta)} = \frac{\sin\theta\cos\theta}{g/\Omega^2R - \sin^2\theta}$$



This procedure can be justified since the sum of the two forces forementioned should lie along the direction of the plumb so that it can be balanced with the tension.

(b) The Lagrangian of this case can be written as, (φ; measured on earth)

$$L = \frac{1}{2}m(\dot r^2 + r^2\sin^2\theta(\dot\varphi+\Omega)^2 + r^2\dot\theta^2) + \frac{GMm}{r}$$
$$= \frac{1}{2}m(\dot r^2 + r^2\sin^2\theta(\dot\varphi^2 + 2\Omega\dot\varphi) + r^2\dot\theta^2) + \left(\frac{GMm}{r} + \frac{1}{2}mr^2\sin^2\theta\,\Omega^2\right)$$

Thus, we can directly see that the effective potential is,

$$V_{eff} = -\frac{GMm}{r} - \frac{1}{2}mr^2\sin^2\theta\,\Omega^2$$

Now we require that the surface of the earth to be the equal potential surface. Thus,

$$-\frac{GMm}{r} - \frac{1}{2}mr^2\sin^2\theta\,\Omega^2 = \text{constant} = -\frac{GMm}{r_p}$$

where we evaluated the constant at $\theta = 0$. Rearranging the above relation gives,

$$r^3\sin^2\theta = \frac{2GM}{\Omega^2}\left(\frac{R}{R_p} - 1\right)$$

Setting $\theta = \pi/2$ corresponds to the choice $r = r_E$. Thus,

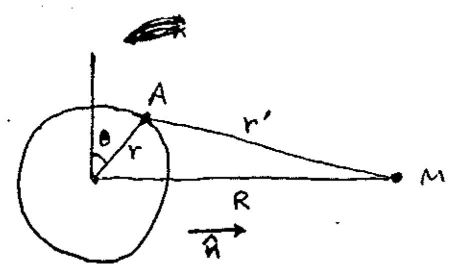
$$\frac{r_E}{r_p} = 1 + \frac{\Omega^2 r_E^3}{2GM} \equiv 1 + \epsilon$$

Consequently,

$$\epsilon = \frac{\Omega^2 r_E^2}{2GM} \simeq 1.7\times10^{-3} = \frac{1}{580}.$$

9. Tidal Bulge

We consider the potential at point A. The potential at this point consists of three parts, namely, the potential due to the earth, potential due to the moon and the fictitious potential due to the rotation of earth about the moon-earth CM. The first part is simply,



$(r'^2 = r^2 + R^2 - 2rR\cos\theta)$

$$V_{earth}/m = -\frac{GM}{r}$$

The second part is,

$$U_{moon}/m = -\frac{GM'}{r'} = -GM'\left(\frac{1}{\sqrt{r^2+R^2-2Rr\cos\theta}}\right) = -\frac{GM'}{R}\left(\frac{1}{\sqrt{1-2r\cos\theta/R + r^2/R^2}}\right)$$
$$= -\frac{GM'}{R}\left(1 - \frac{1}{2}(-2r\cos\theta/R + r^2/R^2) + 3/8\,(-2r\cos\theta/R)^2 + \cdots\right)$$
$$= -\frac{GM'}{R}\left(1 + \frac{r\cos\theta}{R} - \frac{3}{2}\frac{r^2}{R^2}\left(\frac{1}{3} - \cos^2\theta\right) + \cdots\right)$$

where we used Taylor expansion formula, $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$

Since the attractive force due to moon is given by

$$\frac{GMm}{R^2}\hat{n}$$

where $\hat{n}$ is shown in the figure, we need the presence of the counter fictious force,

$$-\frac{GMm}{R^2}\hat{n}$$

in the frame where the earth is at rest. (Otherwise the CM of earth should move)

The effective potential for the above fictitous force is,

$$V_{fictitious}/m = \frac{GM\cancel{m}}{R^2} \times \text{distance along } \hat{n} = \frac{GM}{R^2} r\cos\theta$$

which can be directly verified by the differentiation. Thus, the total potential energy is given by,

$$V/m = -\frac{GM}{r} - \frac{GM'}{R} + \frac{3}{2}\frac{GM'r}{R^3}\left(\frac{1}{3} - \cos^2\theta\right)$$

Let r_0 denote the radius of the earth along the certain θ_c which satisfies $\cos^2\theta_c = \frac{1}{3}$.

As in the previous problem, we require the effective potential to be a constant. Then,

$$-\frac{GM}{r} - \frac{GM'}{R} + \frac{3}{2}\frac{GM'}{R^3}r^2\left(\frac{1}{3} - \cos^2\theta\right) = -\frac{GM}{r_0} - \frac{GM'}{R} = \text{constant}$$

$$\frac{r}{r_0} = \left(1 + \frac{3}{2}\frac{M'r^3}{R^3M}\left(\cos^2\theta - \frac{1}{3}\right)\right) \equiv \left(1 + \epsilon\left(\cos^2\theta - \frac{1}{3}\right)\right) \qquad (1)$$

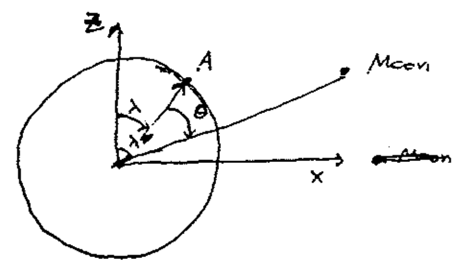
where we evaluated the constant near $\theta = \theta_c$. Thus,

$$\epsilon = \frac{3}{2}\frac{M'}{M}\frac{r^3}{R^3}. \quad \text{(We can assume this is constant approximately)}$$

Clearly, r_0 can be identified as a mean radius as the below calculation shows.

$$\langle r\rangle = r_0 + \frac{\epsilon}{2}\int_0^\pi \sin\theta d\theta\left(\cos^2\theta - \frac{1}{3}\right) = r_0$$

If we set our coordinate system as shown right, the Cartesian components of the angular coordinates are given by,



Moon; $(\sin\lambda', 0, \cos\lambda')$

A ; $(\sin\lambda\cos(\varphi-\varphi'), \sin\lambda\sin(\varphi-\varphi'), \cos\lambda)$

on a unit sphere. The value $\cos\theta$ is exactly the inner product of these two vectors. Thus,

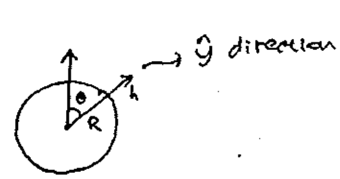
$$\cos\theta = \sin\lambda'\sin\lambda\cos(\varphi-\varphi') + \cos\lambda'\cos\lambda$$

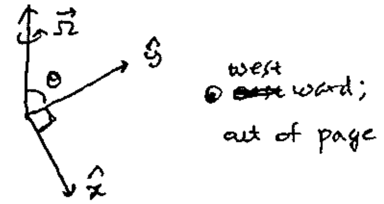
We put this expression into (1) and use $\cos 2\bar{\theta} = 2\cos^2\bar{\theta} - 1$, $\sin 2\bar{\theta} = 2\sin\bar{\theta}\cos\bar{\theta}$, $\cdots$ etc.

Then, we get, (after a little bit of algebra)

$$\frac{r}{r_0} = 1 + \epsilon\left\{\frac{3}{2}(\cos^2\lambda - 1)(\cos^2\lambda' - 1) + \frac{1}{2}\sin^2\lambda\sin^2\lambda'\cos(2\phi - \phi') + \frac{1}{2}\sin 2\lambda\sin 2\lambda'\cos(\phi - \phi')\right\}$$

The periods of each tides are one month, one-half day, and one day, respectively. The second part corresponds to the usual tide which occurs twice a day. Notice that only this part has positive definite amplitude.





10. (a) Since we are considering first order in Ω we can neglect the centrifugal force. If we choose the coordinate system shown right, we can calculate Coriolis force as follows.

$$-2m\vec\Omega\times\vec v = -2m\begin{vmatrix}\hat x & \hat y & \hat z\\ -\Omega\sin\theta & \Omega\cos\theta & 0\\ v_x & v_y & v_z\end{vmatrix} = 2m\Omega v_y\sin\theta\,\hat z \qquad (1)$$

Notice that we neglected the contribution from v_x and v_z since they would be proportional to Ω and we are considering only upto the linear order.Thus, the equations of motion are

$$m\ddot y = -mg \Rightarrow \ddot y = -g, \qquad \ddot x = 0$$
$$m\ddot z = 2m\Omega v_y\sin\theta \Rightarrow \ddot z = 2\Omega v_y\sin\theta$$

From y-component equation, we get $y = h - 1/2\, gt^2$, We put this expression into z-component equation and integrate it twice keeping the initial conditions $z=\dot z = 0$ at $t = 0$. We get,

$$z = \int_0^t\int_0^{t'} 2\Omega(-gt'')\sin\theta\, dt''\, dt' = -\frac{g}{3}\Omega\sin\theta\, t^3$$

Inserting $t = \sqrt{2h/g}$ which is obtained by setting $y = 0$, into the above equation, we get,

$$z = -\frac{2}{3}\sqrt{\frac{2h^3}{g}}\,\Omega\sin\theta$$

Since this quantity is negative (Notice that due to the righthand rule, the positive z direction is west ~~east~~ward), the deflection is eastward.

(b) In this case, from y-component equation we get $y = v_0t - \frac{1}{2}gt^2 = \sqrt{2gh}\,t - \frac{1}{2}gt^2$ We put this expression into z-component equation and integrate it twice keeping the initial conditions. We get,

$$z = \int_0^t\int_0^{t'} 2\Omega\sin\theta(\sqrt{2gh} - gt'')\, dt''\, dt' = \Omega\sin\theta\left(\sqrt{2gh}\,t^2 - \frac{1}{3}gt^3\right)$$

Inserting $t = 2\sqrt{\frac{2h}{g}}$ which is obtained by setting $y = 0$, into the above equation, we get,

$$z = \Omega\sin\theta\cdot\frac{8h}{g}\left(\sqrt{2gh} - \frac{2}{3}\sqrt{2gh}\right) = \frac{8}{3}\sqrt{\frac{2h^3}{g}}\,\Omega\sin\theta$$

Since this quantity is positive, the deflection is westward. Since this result is invariant $\theta \to 180-\theta$ transformation, the result also holds in southern hemisphere. Of course, this conclusion can be drawn from the more detailed analysis of geometry.

(c) From the angular momentum conservation, we have,

$$L = m(R+y)^2\sin^2\theta\,\dot\phi = mR^2\sin^2\theta\,\Omega \Rightarrow \dot\phi = \frac{R^2\Omega}{(R+y)^2} \approx \Omega - 2\frac{y}{R}\Omega$$

Thus, the change in the azimuthal angle when viewed on earth is, ($\Delta\phi$ defined in the problem)

$$\Delta\bar\phi = \int_0^t(\dot\phi - \Omega)\, dt' = \Delta\phi - \Omega t$$
$$= -\frac{2}{R}\Omega\int_0^t y\, dt' = -\frac{2}{R}\Omega\int_0^t\int_0^{t'} v_y\, dt''\, dt'$$

where the last step is valid since y(0) = 0. Thus, the deflection length is given by

$$z = R\sin\theta\,(-\Delta\phi)$$
$$= \int_0^t \int_0^{t'} 2\Omega \sin\theta\, v_y\, dt''\, dt'$$

(Notice that the negative sign in the first equation is necessary in my convention.) This is exactly the equation which was used in the part (b). Thus, two approach give the identical result. $(\Delta\phi = z/R\sin\theta)$

11. (a) We use the result of problem 10, eq.(1). In this case, both v_x and v_y have the zeroth order term in Ω. Thus, we should retain this contribution in Eq.(1). The equations of motion, then, become,

$$\ddot{z} = 2\Omega\dot{y}\sin\theta + 2\Omega\dot{x}\cos\theta\,, \qquad \ddot{x} = 0$$
$$\ddot{y} = -g$$

x and t component equations are solved by elementary method and the results are

$$x = -Dt/t_0\,, \quad y = \sqrt{2gh}\,t - gt^2/2$$

where t_0 denotes the total flight time, $t_0 = 2\sqrt{2h/g}$. Since at t=0, x = y = z = $\dot{z}$ = 0, we have,

$$\dot{z} = 2\Omega y\sin\theta + 2\Omega x\cos\theta$$
$$z = 2\Omega\int_0^{t_0}\left((\sqrt{2gh}\,t - \tfrac{1}{2}gt^2)\sin\theta - \frac{t}{t_0}D\cos\theta\right)dt = 2\sqrt{\frac{2h}{g}}\,\Omega\left(\frac{4}{3}h\sin\theta - D\cos\theta\right) \quad \left(\begin{matrix} z>0 \\ z<0 \end{matrix}\ \text{case}\right)$$

(b) In this case, both v_y and v_z have the zeroth order term in Ω. Thus, the proper equations of motion are

$$\ddot{x} = -2\Omega\dot{z}\cos\theta\,, \qquad \ddot{z} = 2\Omega\sin\theta\,\dot{y}$$
$$\ddot{y} = -g - 2\Omega\sin\theta\,\dot{z}$$

The zeroth order trajectory is same as above except for the shooting direction. Thus,

$$z^{(0)} = -\frac{t}{t_0^{(0)}}D\,, \quad y^{(0)} = v_0 t - \frac{1}{2}gt^2 \qquad (v_0 = \sqrt{2gh}\,,\ t_0^{(0)} = 2\sqrt{2h/g})\,,\ x^{(0)} = 0$$

Putting z calculated above into y-component equation gives,

$$\ddot{y} = -g + 2\Omega\sin\theta\,\frac{D}{t_0^{(0)}}$$

We integrate above equation twice and the travel time is given by setting y =0. That is

$$t_0 = 2v_0/(g - 2\Omega\sin\theta\, D/t_0^{(0)}) = t_0^{(0)} + \frac{2\Omega}{g}\sin\theta\, D$$

Thus, the position of landing is given by,

$$z = -D\left(1 + \frac{2\Omega}{g}\sin\theta\, D\cdot\frac{1}{t_0^{(0)}}\right) + \int_0^{t_0^{(0)}} 2\Omega\sin\theta\, y\, dt$$

which gives the distance,

$$|z| = D\left(1 + \Omega D\sin\theta/\sqrt{2gh}\right) - \frac{8}{3}\sqrt{\frac{2h}{g}}\,\Omega h\sin\theta$$

The deflected lengthalong x direction is,

$$x^{(1)} = 2\Omega\cos\theta\int_0^{t_0}\frac{D}{t_0}\,t\,dt = t_0 D\Omega\cos\theta = 2\Omega\sqrt{\frac{2h}{g}}\,D\cos\theta$$

which is southward since it is positive. If shooting angle is small, we can drop the second term in (2).