The Force on an Antenna Array

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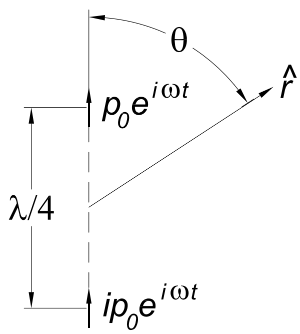
1 Problem

Antennas radiate (vector) momentum as well as energy, but if the radiation pattern is symmetric, the total radiated momentum is zero. Consider a simple antenna array that consists of two short linear dipole antennas whose centers are $\lambda/4$ apart and whose conductors point along the line of centers of the antennas. If the two antennas are excited with a 90° phase difference, deduce the asymmetric radiation pattern and the time-average rate of momentum $d\langle\mathbf{P}\rangle/dt$, that is radiated. The antenna experiences a time-average reaction force $\langle\mathbf{F}\rangle = -d\langle\mathbf{P}\rangle/dt$.[1]



2 Solution

This problem is based on prob. 6, p. 400 of [1].[2] For another application of the concept of the radiation reaction to antennas, see [2].

The momentum density $\mathbf{p}$ and energy density u of electromagnetic radiation that moves in direction $\hat{\mathbf{x}}$ with speed of light c are related by $u = cp$. This suggests that the reaction force will be related to the radiated power dU/dt as,

$$\langle F\rangle = \frac{K}{c}\frac{dU}{dt}\,, \tag{1}$$

where K is a dimensionless constant (possibly zero) dependent on details of the radiated momentum distribution.

[1] (Apr. 21, 2021. Thanks to Jeff Green.) When a receiving antenna detects a signal by extracting energy ΔE in time Δt from a wave incident on the antenna with direction $\hat{\mathbf{k}}$, then there is a force $\mathbf{F} = \Delta\mathbf{P}/\Delta t = \Delta E\,\hat{\mathbf{k}}/c\Delta T$ on the antenna (in Gaussian units), where c is the speed of light in vacuum. In general, there is an additional force on the antenna associated with the scattering (rather than absorption) of the incident wave. The mechanical force on the antenna is not measured, but the electrical signal is proportional to the part of the mechanical force associated with the absorbed signal energy ΔE.

[2] This paper is dedicated to the memory of W.K.H. "Pief" Panofsky, who passed away on Sept. 24, 2007 at age 88.

The time-average pattern of radiated energy, $d\langle U\rangle/d\Omega\,dt$, of an antenna with electric dipole moment $\mathbf{p}$ can be deduced from the radiation fields in the far zone via the time-average Poynting vector,

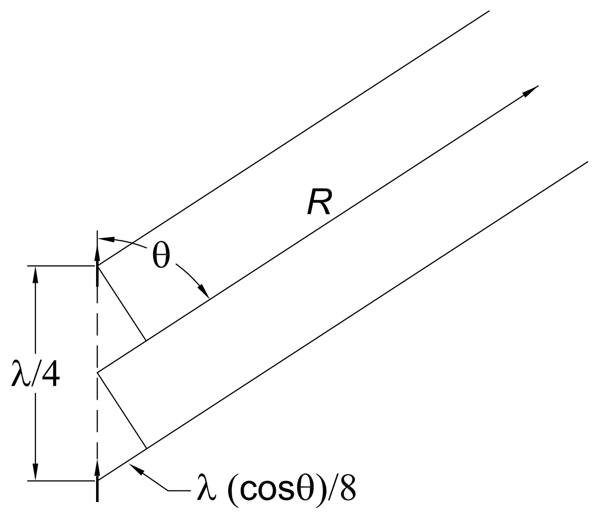
$$\langle \mathbf{S}_{\rm rad}\rangle = \frac{c}{8\pi} Re(\mathbf{E}_{\rm rad} \times \mathbf{B}^\star_{\rm rad}), \tag{2}$$

in Gaussian units, according to,

$$\frac{d\langle U\rangle}{d\Omega\,dt} = r^2 \langle \mathbf{S}_{\rm rad}\rangle \cdot \hat{\mathbf{r}} = \frac{|[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}|^2}{8\pi c^3}, \tag{3}$$

where $[\ddot{\mathbf{p}}] = \ddot{\mathbf{p}}(t' = t - r/c)$ is the second time derivative of the dipole moment evaluated at the retarded time, $t' = t - r/c$ for an observer at distance r from the dipole.

In the present example the dipole moment is the sum of two terms, $p_0\, e^{i\omega t}\,\hat{\mathbf{z}}$ due to a small antenna at $z = \lambda/8$, and $ip_0\, e^{i\omega t}\,\hat{\mathbf{z}}$ due to a second antenna at $z = -\lambda/8$ that is driven 90° out of phase with respect to the first. If the distance from the center of the array to the observer is R, then the distance to the upper antenna is $R - \lambda(\cos\theta)/8$ while the distance to the lower antenna is $R + \lambda(\cos\theta)/8$, as sketched in the figure below.



The retarded derivative of the total dipole moment is therefore,

$$\begin{aligned} [\ddot{\mathbf{p}}] &= -\omega^2 p_0\, e^{i\omega(t-R/c)} \left(e^{i\omega\lambda(\cos\theta)/8c} + i\, e^{-i\omega\lambda(\cos\theta)/8c}\right) \hat{\mathbf{z}} \\ &= -\omega^2 p_0\, e^{i\omega(t-R/c)}\, e^{i\pi(\cos\theta)/4} \left(1 + i\, e^{-i\pi(\cos\theta)/2}\right) \hat{\mathbf{z}}. \end{aligned} \tag{4}$$

Then,

$$[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}} = -\omega^2 p_0\, e^{i\omega(t-R/c)}\, e^{i\pi(\cos\theta)/4} \left(1 + i\, e^{-i\pi(\cos\theta)/2}\right) \sin\theta\, \hat{\boldsymbol{\phi}}, \tag{5}$$

$$|[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}|^2 = 2\omega^4 p_0^2 \sin^2\theta \left(1 + \sin\frac{\pi}{2}\cos\theta\right), \tag{6}$$

and,

$$\frac{d\langle U\rangle}{d\Omega\,dt} = \frac{|[\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}|^2}{8\pi c^3} = \frac{\omega^4 p_0^2 \sin^2\theta}{4\pi c^3} \left(1 + \sin\frac{\pi}{2}\cos\theta\right). \tag{7}$$

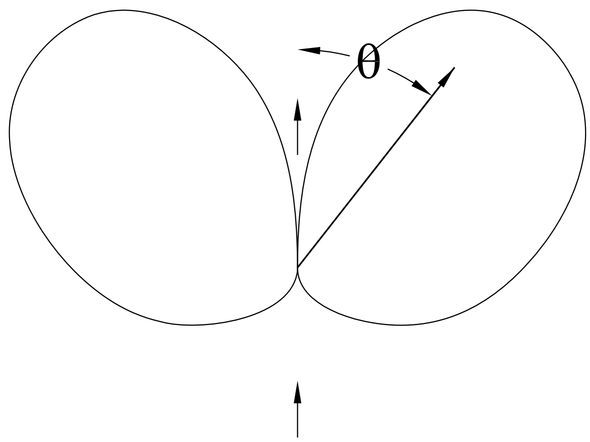
This angular distribution is sketched in the figure on the next page. More energy is radiated into the forward hemisphere ($z > 0$) than the backward.

The total radiated energy is obtained by integration of eq. (7) over solid angle,

$$\frac{d\langle U\rangle}{dt} = \int \frac{d\langle U\rangle}{d\Omega\, dt}\, d\Omega = \frac{2\omega^4 p_0^2}{3c^3}\,. \tag{8}$$

Associated with the radial flow of energy $\langle U\rangle$ in the far zone is a radial flow of momentum,[3] $\langle \mathbf{P}\rangle = \langle U\rangle\, \hat{\mathbf{r}}/c$. Hence, the angular distribution of time-average momentum radiated by the antenna follows from eq. (7) as,

$$\frac{d\langle \mathbf{P}\rangle}{d\Omega\, dt} = \frac{\omega^4 p_0^2 \sin^2\theta}{4\pi c^4}\left(1 + \sin\frac{\pi}{2}\cos\theta\right)\hat{\mathbf{r}}. \tag{9}$$



On integrating this over solid angle to find the total momentum radiated, only the z component is nonzero,

$$\begin{aligned}
\frac{d\langle \mathbf{P}\rangle_z}{dt} &= 2\pi\frac{\omega^4 p_0^2}{4\pi c^4}\int_{-1}^{1} \sin^2\theta\left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right]\cos\theta\; d\cos\theta \\
&= \frac{4\omega^4 p_0^2}{\pi^2 c^4}\int_0^{\pi/2} x\left(1 - \frac{4x^2}{\pi^2}\right)\sin x\; dx \qquad \text{where } x = \frac{\pi}{2}\cos\theta \\
&= \frac{8\omega^4 p_0^2}{\pi^2 c^4}\left(\frac{12}{\pi^2} - 1\right) = \frac{12}{\pi^2 c}\frac{d\langle U\rangle}{dt}\left(\frac{12}{\pi^2} - 1\right) \approx \frac{0.26}{c}\frac{d\langle U\rangle}{dt}\,.
\end{aligned} \tag{10}$$

The radiation reaction force on the antenna is $F_z = -dP_z/dt$. For a broadcast antenna radiating 10^5 Watts, the reaction force would be only $\approx 10^{-4}$ N.

It is useful to re-express the dipole moment in terms of the current distribution in the antennas. To a good first approximation the latter can be written for each of the two antennas as the triangular form,

$$I_j(z_j, t) = I_0\left(1 - \frac{2\,|z_j|}{L}\right) e^{i(\omega t + \phi_j)}, \tag{11}$$

[3]This is the classical version of the quantum relation for photons that $U = \hbar\omega$ and $\mathbf{P} = \hbar\,\mathbf{k} = \hbar\omega\,\hat{\mathbf{k}}/c = U\,\hat{\mathbf{k}}/c$.

where $j = 1, 2$. The current vanishes at the tips of the antenna, at $z_j = \pm |L|/2$, and has a peak value of I_0 at the feedpoint at $z_j = 0$.[4] The associated charge distributions $\rho(z_j, t)$ are related by current conservation, $\boldsymbol{\nabla} \cdot \mathbf{J} = -\dot{\rho}$, which for a 1-D distribution is simply.

$$\dot{\rho}_j = i\omega\rho_j = -\frac{\partial I_j}{\partial z_j} = -I_0 \left(\mp \frac{2}{L} \right) e^{i(\omega t + \phi_j)}, \tag{12}$$

where the upper sign is for $z_j > 0$ and the lower for $z_j < 0$, so that,

$$\rho_j = \pm \frac{2I_0}{i\omega L} e^{i(\omega t + \phi_j)}. \tag{13}$$

The dipole moment is given by,

$$p_j = \int_{-L/2}^{L/2} \rho_j z_j \, dz_j = \frac{I_0 L}{2i\omega} e^{i(\omega t + \phi_j)}. \tag{14}$$

Thus, the magnitude p_0 of the dipole moment of the antenna is related to the peak current I_0 by,

$$p_0 = \frac{I_0 L}{2\omega}, \tag{15}$$

Using eq. (15) in eqs. (7)-(10) and noting that $\omega = 2\pi c/\lambda$, the radiated power can be written as,

$$\frac{d\langle U \rangle}{dt} = \frac{I_0^2}{2} \frac{4\pi^2}{3c} \frac{L^2}{\lambda^2} = \frac{I_0^2 R_{\rm rad}}{2}, \tag{16}$$

where (using $1/c = 30\,\Omega$),

$$R_{\rm rad} = \frac{4\pi^2}{3c} = 395\,\Omega, \tag{17}$$

is the radiation resistance of the antenna, and the reaction force on the antenna is,

$$\langle F_z \rangle = -\frac{d\langle \mathbf{P} \rangle_z}{dt} = -\frac{8I_0^2 L^2}{c^2 \lambda^2} \left(\frac{12}{\pi^2} - 1 \right) = -\frac{12}{\pi^2 c} \frac{d\langle U \rangle}{dt} \left(\frac{12}{\pi^2} - 1 \right) \approx -\frac{0.26}{c} \frac{d\langle U \rangle}{dt}. \tag{18}$$

Recalling eq. (1), the magnitude of constant K is 0.26 for the present example.

The radiation reaction force (18) cannot, in general, be deduced as the sum over charges of the radiation reaction force of Lorentz [4], $\mathbf{F}_{\rm rad} = 2e^2\ddot{\mathbf{v}}/3c^3$. Lorentz' result is obtained by an integration by parts of the integral of the radiated power over a period.[5] This procedure can be carried out if the power is a sum/integral of a square, as holds for an isolated radiating charge. But it cannot be carried out when the power is the square of a sum/integral as holds for (coherent) radiation by an extended charge/current distribution. Rather, the radiation

[4]The current distribution (11) does not satisfy the metallic boundary condition that the electric field have zero tangential component at the surface of the conductors. However, eq. (11) suffices for a good understanding of the far-zone radiation pattern. What is lost in the approximation (11) is the relation between voltage and current at the antenna feedpoint, *i.e.*, the impedance of the antenna. For further discussion see [3].

[5]This now-familiar derivation of $\mathbf{F}_{\rm rad}$ was first given by Planck [5] rather by Lorentz. See also sec. 13 of [6].

reaction force on an extended current distribution must be deduced from the rate of radiation of momentum, as done here.[6,7]

References

[1] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2$^{\rm nd}$ ed. (Addison-Wesley, 1962),
http://kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[2] K.T. McDonald, *The Radiation Reaction Force and the Radiation Resistance of Small Antennas* (Jan. 21, 2006), http://kirkmcd.princeton.edu/examples/resistance.pdf

[3] K.T. McDonald, *Currents in a Center-Fed Linear Dipole Antenna*, (June 27, 2007), http://kirkmcd.princeton.edu/examples/transmitter.pdf

[4] H.A. Lorentz, *La Théorie Électromagnétique de Maxwell et son Application aux Corps Mouvants*, Arch. Neérl. **25**, 363-552 (1892), reprinted in *Collected Papers* (Martinus Nijhoff, The Hague, 1936), Vol. II, pp. 64-343,
http://kirkmcd.princeton.edu/examples/EM/lorentz_theorie_electromagnetique_92.pdf
See also sec. 27 and note 16 of *The Theory of Electrons* (Teubner, Leipzig, 1909),
http://kirkmcd.princeton.edu/examples/EM/lorentz_theory_of_electrons_09.pdf

[5] M. Planck, *Über electrische Schwingungen, welche durch Resonanz erregt und durch Strahlung gedämpft werden*, Sitz. K. P. Akad. Wiss. Berlin, 151 (1896),
http://kirkmcd.princeton.edu/examples/EM/planck_skpaw_151_96.pdf
Ann. d. Phys. **60**, 577 (1897),
http://kirkmcd.princeton.edu/examples/EM/planck_ap_60_577_97.pdf

[6] K.T. McDonald, *On the History of the Radiation Reaction* (May 6, 2017),
http://kirkmcd.princeton.edu/examples/selfforce.pdf

[7] K.T. McDonald, *Tuval's Electromagnetic Spaceship* (Nov. 30, 2015),
http://kirkmcd.princeton.edu/examples/tuval.pdf

[6] The radiation reaction force (18) of Lorentz is still useful for macroscopic current distributions whose extent is small compared to the wavelength of the emitted radiation. See [2].

[7] An antenna that emits met momentum can be considered as a kind of "electromagnetic rocket." The momentum transfer to the antenna can also be computed directly by integrating the Lorentz force over the antenna. See, for example, [7].