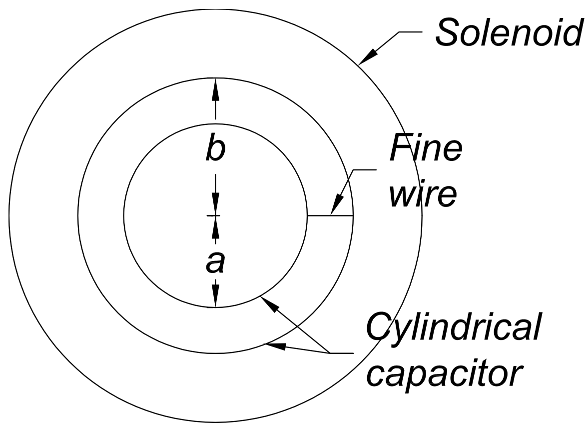
The Barnett Experiment with a Rotating Solenoid Magnet

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 6, 2003; updated August 10, 2008)

1 Problem

A cylindrical capacitor of inner and outer radii a and b is coaxial with a solenoid magnet of radius greater than b, as shown in the figure below. The magnetic field of the solenoid has magnitude B in the lab frame. A fine wire shorts the inner and outer conductors of the capacitor, as shown.



Deduce the charge Q per unit length along the axis that would be observed on the outer cylinder of the capacitor in the following two cases:

1. The capacitor is rotating about its axis in the lab frame with angular velocity ω, while the solenoid is at rest in the lab frame.

2. The capacitor is at rest in the lab frame, but the solenoid is rotating with angular velocity ω about its axis in the lab frame.

In each case, analyze the problem both in the lab frame and in a frame that rotates with angular velocity ω. You may assume that $\omega b \ll c$ where c is the speed of light, and that any magnetic field due to the rotation of the capacitor is negligible.

2 Solution

This problem is based on an experiment by Barnett [1] that was meant to address the question of whether the magnetic field lines of a solenoid magnet rotate when the magnet rotates about its axis. See also [2]. It will also illustrate a remark by Schiff [3] that care must be taken when discussing electrodynamics in rotating frames, because the effect of the "distant stars" is to introduce "fictitious" charges and currents into the problem.

2.1 The Capacitor Rotates in the Lab Frame

2.1.1 Analysis in the Lab Frame

As the fine wire of the capacitor rotates through the magnetic field B of the solenoid, the Lorentz force causes an effective radial electric field,

$$E_{r,\text{eff}} = \frac{v}{c} B = \frac{\omega r B}{c} \,, \tag{1}$$

in Gaussian units. As a result, there is a voltage difference across the capacitor given by,

$$V = \int_a^b E_{r,\text{eff}} \; dr = \frac{\omega B}{c} (b^2 - a^2). \tag{2}$$

Since the capacitance per unit length of the cylindrical capacitor is,

$$C = \frac{1}{2 \ln(b/a)} \,, \tag{3}$$

the charge per unit length on the capacitor electrodes is,

$$Q = CV = \frac{\omega B (b^2 - a^2)}{2c \, \ln(b/a)} \,. \tag{4}$$

2.1.2 Analysis in the Rotating Frame

According to eq. (9) of the Appendix, the magnetic field $\mathbf{B}'$ observed in the rotating frame is the same as the magnetic field $\mathbf{B}$ in the lab frame, and the electric field $\mathbf{E}'$ observed in the rotating frame is given by,

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{5}$$

where $\mathbf{v}$ is the lab-frame velocity of the point of observation, which is at rest in the rotating frame. The term $\mathbf{v}/c \times \mathbf{B}$ in $\mathbf{E}'$ is due to the "fictitious" charge density $\boldsymbol{\omega} \cdot \mathbf{B}'/2\pi c = \boldsymbol{\omega} \cdot \mathbf{B}/2\pi c$ that appears in the Maxwell equation (21) in the rotating frame. This leads to a radial electric field in the rotating frame,

$$E'_{r,\text{eff}} = \frac{v}{c} B = \frac{\omega r B}{c} = E_{r,\text{eff}} \,, \tag{6}$$

so the rest of the analysis in the rotating frame is identical to that in sec. 2.2.1 for the lab frame.

2.2 The Solenoid Rotates in the Lab Frame

2.2.1 Analysis in the Lab Frame

In the second experiment, the capacitor is at rest in the lab frame, so there is no $\mathbf{v} \times \mathbf{B}$ force on the charges in the capacitor, and hence no charge separation. Thus, we predict that $Q = 0$, as was confirmed by Barnett in his experiment [1].

There is an interpretation of Faraday's law of induction that $\mathcal{EMF}$'s are generated when part or all of a circuit cuts magnetic field lines. By this hypothesis, if the magnetic field lines were rotating together with the solenoid, then the fine wire, although at rest, would be cutting field lines, and an $\mathcal{EMF}$ should be generated, and a charge separation detected on the capacitor. Since no such separation occurs, one can conclude that magnetic field lines do not rotate together with the solenoid.

The original motivation of Barnett's experiment [1] appears to have been to confirm the above interpretation.

2.2.2 Analysis in the Rotating Frame

The solenoid is at rest in the rotating frame, but the capacitor has angular velocity $-\omega$ here.

We again neglect any possible effect of the capacitor on the magnetic field, so again magnetic field in the rotating frame $\mathbf{B}'$ is that same as the lab-frame field $\mathbf{B}$ of the solenoid. Again, there appears to be an effective ("fictitious") charge density $\rho'_{\text{eff}} = \omega B/2\pi c$ inside the solenoid, which implies a radial electric field (in the rotating frame) of magnitude $\omega r B/c$.

The important difference between this case and the previous is that now the capacitor and the fine wire are rotating with angular velocity $-\omega$ in the rotating frame. Hence, there is a $\mathbf{v}'/c \times \mathbf{B}' = -\mathbf{v}/c \times \mathbf{B}$ force on charges in the capacitor,[1] as if there is an additional radial electric field of magnitude $\omega r B/c$. The direction of this effective field is opposite to that caused by the "fictitious" charge distribution (because the angular velocity of the capacitor in the rotating frame is the negative of the angular velocity of the rotating frame with respect to the lab frame). Thus, these two radial components of the electric field cancel in the rotating frame.

The inner and outer conductors of the cylindrical capacitor can now be at the same potential with no charge separation. That is, we predict that $Q = 0$ from the analysis in the rotating frame, in agreement with the analysis in the lab frame.

A Summary of the Principles of Electrodynamics in a Rotating Frame

For reference, we reproduce the principles of electrodynamics in the frame of a slowly rotating medium where ϵ and μ differ from unity.[2,3]

The (cylindrical) coordinate transformation is,

$$r' = r, \qquad \phi' = \phi - \omega t, \qquad z' = z, \qquad t' = t, \tag{7}$$

where quantities in observed in the rotating frame are labeled with a $'$. The transformations of free charge and current density are,

$$\rho'_{\text{free}} = \rho_{\text{free}}, \qquad \mathbf{J}'_{\text{free}} = \mathbf{J}_{\text{free}} - \rho_{\text{free}}\,\mathbf{v}, \tag{8}$$

[1]As noted in eq. (9), the Lorentz force law has the same form in the rotating frame as in the lab frame.

[2]This Appendix is from sec. 2.2.5 of [4].

[3]This case is discussed most thoroughly by Ridgely [5, 6], but primarily for the interesting limit of steady charge and current distributions.

where $\mathbf{v}$ (with $v = \omega r \ll c$) is the velocity with respect to the lab frame of the observer in the rotating frame at distance $r' = r$ from its axis. The transformations of the electromagnetic fields are,

$$\mathbf{B}' = \mathbf{B}, \qquad \mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{H}' = \mathbf{H}. \tag{9}$$

The transformations of the electric and magnetic polarizations are,

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \qquad \mathbf{M}' = \mathbf{M}, \tag{10}$$

if we regard these polarizations as defined by $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{M}'$.

The lab-frame bound charge and current densities $\rho_{\rm bound} = -\boldsymbol{\nabla}\cdot\mathbf{P}$ and $\mathbf{J}_{\rm bound} = \partial\mathbf{P}/\partial t + c\,\boldsymbol{\nabla} \times \mathbf{M}$ transform to,

$$\rho'_{\rm bound} = -\boldsymbol{\nabla}'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}', \tag{11}$$

$$\mathbf{J}'_{\rm bound} = \frac{\partial\mathbf{P}'}{\partial t'} + c\,\boldsymbol{\nabla}'\times\mathbf{M}' + \mathbf{v}(\boldsymbol{\nabla}'\cdot\mathbf{P}') + \frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{M}'}{\partial t'} + (\mathbf{P}'\cdot\boldsymbol{\nabla})\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{P}'. \tag{12}$$

Force $\mathbf{F}$ is invariant under the transformation (7). In particular, a charge q with velocity $\mathbf{v}_q$ in the lab frame experiences a Lorentz force in the rotating frame given by,

$$\mathbf{F}' = q\left(\mathbf{E}' + \frac{\mathbf{v}'_q}{c}\times\mathbf{B}'\right) = q\left(\mathbf{E} + \frac{\mathbf{v}_q}{c}\times\mathbf{B}\right) = \mathbf{F}, \tag{13}$$

where $\mathbf{v}'_q = \mathbf{v}_q - \mathbf{v}$. Similarly, the Lorentz force density $\mathbf{f}'$ on charge and current densities in the rotating frame is,

$$\mathbf{f}' = \rho'\mathbf{E}' + \frac{\mathbf{J}'}{c}\times\mathbf{B}' = (\rho'_{\rm free} + \rho'_{\rm bound})\,\mathbf{E}' + \frac{\mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound}}{c}\times\mathbf{B}'. \tag{14}$$

Maxwell's equations in the rotating frame can be written as,

$$\boldsymbol{\nabla}'\cdot\mathbf{B}' = 0, \tag{15}$$

$$\boldsymbol{\nabla}'\cdot\mathbf{D}' = 4\pi\rho'_{\rm free,total} = 4\pi\left(\rho'_{\rm free} + \rho'_{\rm other}\right), \tag{16}$$

$$\boldsymbol{\nabla}'\times\mathbf{E}' + \frac{\partial\mathbf{B}'}{\partial ct'} = 0, \tag{17}$$

$$\boldsymbol{\nabla}'\times\mathbf{H}' - \frac{\partial\mathbf{D}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm free,total} = \frac{4\pi}{c}\left(\mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm other}\right), \tag{18}$$

where $\rho'_{\rm free} = \rho_{\rm free}$ and $\mathbf{J}'_{\rm free} = \mathbf{J}_{\rm free} - \rho_{\rm free}\mathbf{v}$ are the free charge and current densities, and the "other" charge and current densities that appear to an observer in the rotating frame are,

$$\rho'_{\rm other} = -\frac{\mathbf{v}\cdot\mathbf{J}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial\mathbf{D}'}{\partial ct'}, \tag{19}$$

$$\mathbf{J}'_{\rm other} = \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi}\cdot\boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{H}'}{\partial t'}. \tag{20}$$

The "other" charge and current distributions are sometimes called "fictitious" [3], but we find this term ambiguous. For an example with an "other" charge density $\boldsymbol{\omega} \cdot \mathbf{H}'/2\pi c$ in the rotating frame, see [1].

Maxwell's equations can also be expressed only in terms of the fields $\mathbf{E}'$ and $\mathbf{B}'$ and charge and current densities associated with free charges as well as with electric and magnetic polarization,

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi\rho'_{\rm total}, \tag{21}$$

and,

$$\boldsymbol{\nabla}' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm total}, \tag{22}$$

where,

$$\begin{aligned}
\rho'_{\rm total} &= \rho'_{\rm free} - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}'_{\rm free} - \boldsymbol{\nabla}' \cdot \mathbf{P}' + \frac{\boldsymbol{\omega}' \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{D}'}{\partial ct'} \\
&= \rho'_{\rm free,total} - \boldsymbol{\nabla}' \cdot \mathbf{P}' \\
&= \rho'_{\rm free} + \rho'_{\rm bound} + \rho'_{\rm more}\,, \qquad (23) \\
\rho'_{\rm more} &= -\frac{\mathbf{v}}{c^2} \cdot \left(\mathbf{J}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}' \times \mathbf{M}'\right) + \frac{\boldsymbol{\omega}' \cdot \mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{E}'}{\partial ct'}\,, \qquad (24) \\
\mathbf{J}'_{\rm total} &= \mathbf{J}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}' \times \mathbf{M}' + \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi} \cdot \boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'} \\
&= \mathbf{J}'_{\rm free,total} + \frac{\partial \mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}' \times \mathbf{M}' \\
&= \mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound} + \mathbf{J}'_{\rm more}\,, \qquad (25) \\
\mathbf{J}'_{\rm more} &= \mathbf{v}\left(\rho'_{\rm free} - \boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega}' \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{M}'\right) \\
&\qquad + \left(\frac{\mathbf{E}'}{4\pi} \cdot \boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\frac{\mathbf{E}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{B}'}{\partial t'}\,. \qquad (26)
\end{aligned}$$

The contribution of the polarization densities to the source terms in Maxwell's equations in much more complex in the rotating frame than in the lab frame. Because of the "other" source terms that depend on the fields in the rotating frame, Maxwell's equations cannot be solved directly in this frame. Rather, an iterative approach is required in general.

The constitutive equations for linear isotropic media at rest in the rotating frame are,

$$\mathbf{D}' = \epsilon\mathbf{E}', \qquad \mathbf{B}' = \mu\mathbf{H}' - (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}', \tag{27}$$

in the rotating frame, and,

$$\mathbf{D} = \epsilon\mathbf{E} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{B} = \mu\mathbf{H} - (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{28}$$

in the lab frame. The lab-frame constitutive equations (28) are the same as for a nonrotating medium that moves with constant velocity $\mathbf{v}$ with respect to the lab frame.

We can also write the constitutive equations (27) for a linear isotropic medium in terms of the fields $\mathbf{B}'$, $\mathbf{E}'$, $\mathbf{P}'$ and $\mathbf{M}'$ by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that,

$$\begin{aligned} \mathbf{P}' &= \frac{\epsilon - 1}{4\pi}\mathbf{E}', \\ \mathbf{M}' &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}'. \end{aligned} \tag{29}$$

Similarly, the constitutive equations (28) in the lab frame can be written to order v/c as,

$$\begin{aligned} \mathbf{P} &= \frac{\epsilon - 1}{4\pi}\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{B}}{4\pi} = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \frac{\epsilon\mu - 1}{\mu - 1}\frac{\mathbf{v}}{c} \times \mathbf{M}, \\ \mathbf{M} &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}. \end{aligned} \tag{30}$$

Ohm's law for the conduction current $\mathbf{J}_C$ has the same form for a medium with velocity $\mathbf{u}'$ relative to the rotating frame as it does for a medium with velocity $\mathbf{u}$ relative to the lab frame,

$$\mathbf{J}'_C = \sigma\left(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}'\right) = \sigma\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{J}_C, \tag{31}$$

where σ is the electric conductivity of a medium at rest.

References

[1] S.J. Barnett, *On Electromagnetic Induction and Relative Motion*, Phys. Rev. **35**, 323 (1912), http://kirkmcd.princeton.edu/examples/EM/barnett_pr_35_323_12.pdf

[2] W. Cramp and E.H. Norgrove, *Some Investigations on the Axial Spin of a Magnet and on the Laws of Electromagnetic Induction*, J. IEE **78**, 481 (1936), http://kirkmcd.princeton.edu/examples/EM/cramp_jiee_78_481_36.pdf

[3] L.I. Schiff, *A Question in General Relativity*, Proc. Nat. Acad. Sci. **25**, 391 (1939), http://kirkmcd.princeton.edu/examples/EM/schiff_proc_nat_acad_sci_25_391_39.pdf

[4] K.T. McDonald *Electrodynamics of Rotating Systems* (Aug. 6, 2008), http://kirkmcd.princeton.edu/examples/rotatingEM.pdf

[5] C.T. Ridgely, *Applying relativistic electrodynamics to a rotating material medium*, Am. J. Phys. **66**, 114 (1998), http://kirkmcd.princeton.edu/examples/EM/ridgely_ajp_66_114_98.pdf

[6] C.T. Ridgely, *Applying covariant versus contravariant electromagnetic tensors to rotating media*, Am. J. Phys. **67**, 414 (1999), http://kirkmcd.princeton.edu/examples/EM/ridgely_ajp_67_414_99.pdf