Nahin's Bobsled

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1 Problem

In the Preface to *In Praise of Simple Physics*, Paul Nahin posed the problem illustrated in the figure below.[1] The initial velocity of bobsled B is actually along the downward slope, so that the bobsled is always in contact with the slope.[2] There is assumed to be no friction in both cases A and B.

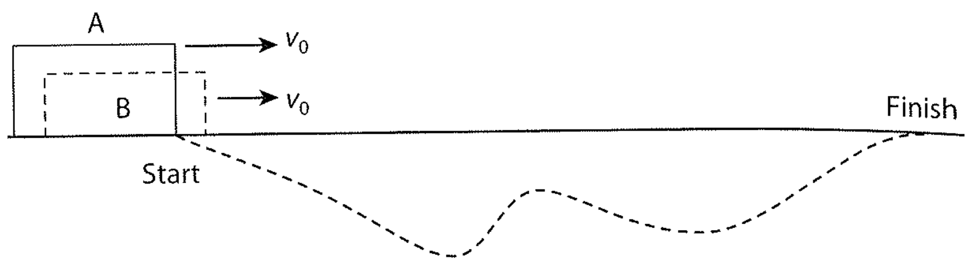


Figure P4. Which bobsledder wins the race?

In the solution offered by Nahin it is claimed that bobsled B always wins the race. Can this be so?

[1] http://kirkmcd.princeton.edu/examples/mechanics/nahin_bobsled.pdf

[2] If the nominal path for bobsled B had a portion with a negative second derivative (as in the figure), the bobsled would lose contact with the slope for high enough initial velocity. Hence, the problem would be better posed by adding the requirement that the path be everywhere concave upwards (nonnegative second derivative everywhere). Or, the problem could be posed for beads sliding without friction on wires.

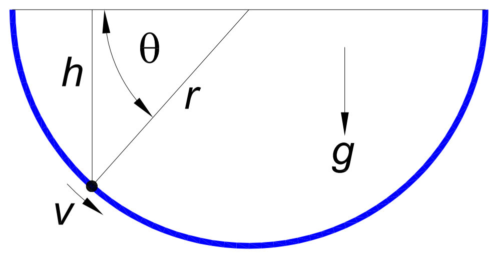
2 Solution

In the low-velocity limit, bobsled B does win the race, as bobsled A moves very slowly on its horizontal path, while bobsled B gets a velocity boost from gravity.[3] However, in the high-velocity limit, gravity has little/no effect on the velocity of bobsled B, which remains $\approx v_0$ at all times. Then, since path B is longer than path A, bobsled B loses the race.

There is some intermediate value of v_0 for which the race is a tie. By dimensional analysis, this velocity is of order $\sqrt{gh}$, where h is the distance between the horizontal path A and the lowest point on path B.

2.1 Skateboarding a Half Pipe

We illustrate the above argument for the case where path B is a semicircle (half pipe) of radius r.



Then, the travel time of bobsled/skateboard A is simply,

$$t_A = \frac{2r}{v_0}\,. \tag{1}$$

As bobsled/skateboard B slides on the half pipe, its velocity when its position is at angle θ to the horizontal is,

$$v(\theta) = \sqrt{v_0^2 + 2gh} = \sqrt{v_0^2 + 2gr\sin\theta} = r\,\frac{d\theta}{dt}\,, \tag{2}$$

as follows from conservation of energy for zero friction. The travel time of bobsled/skateboard B is,

$$t_B = \int dt = \int_0^\pi d\theta\,\frac{dt}{d\theta} = r\int_0^\pi \frac{d\theta}{\sqrt{v_0^2 + 2gr\sin\theta}} \tag{3}$$

For $v_0^2 \gg 2gr$, we have that,

$$t_B \approx \frac{r}{v_0}\int_0^\pi d\theta = \frac{\pi r}{v_0} > t_A \qquad \left(v_0 \gg \sqrt{2gr}\right). \tag{4}$$

[3]In the presence of friction, the outcome can be different. For example, with sliding friction described by $F = \mu N$, where N is the magnitude of the normal force on the sled, if v_0 is just sufficient that sled A finishes the race, sled B will not finish. This is because the normal force on B (for concave-upward paths) is greater than that on A, and path B is longer than path A, such that the loss of energy to friction will be greater for B than for A.

In contrast, for $v_0 = 0$,

$$t_B = \sqrt{\frac{r}{2g}} \int_0^\pi \frac{d\theta}{\sqrt{\sin\theta}} \approx 5.24\sqrt{\frac{r}{2g}} < t_A \ (=\infty) \qquad (v_0 = 0)\,, \tag{5}$$

using Wolfram Alpha to evaluate the integral.

There is an intermediate value of v_0 such that the travel times of bobsled/skateboard A and B are equal,

$$t_B = r\int_0^\pi \frac{d\theta}{\sqrt{v_0^2 + 2gr\sin\theta}} = t_A = \frac{2r}{v_0}\,, \qquad \Rightarrow \qquad \int_0^{\pi/2} \frac{d\theta}{\sqrt{\frac{v_0^2}{2gr} + \sin\theta}} = \sqrt{\frac{2gr}{v_0^2}}\,. \tag{6}$$

Using Wolfram Alpha, we find that $v_0 \approx 0.61\sqrt{2gr} = 0.86\sqrt{gr}$ to have $t_A = t_B$.