Maxwell
and the Development of Electromagnetic Theory

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Abstract

This is a transcription of a typewritten note [1] by Alfred M. Bork, dated July 20, 1976, found in Box 26 of the Bork Archives [2] at UC Irvine.

It is likely this is the manuscript mentioned in Ref. 11 of [3] (1967), and in the Preface of [4] (1968), which was meant to extend the discussion in two papers of Bork from 1967 [5, 6].

The original manuscript had no footnotes, references or figure captions.

Maxwell's *Treatise on Electricity and Magnetism* [7, 8] was first published in 1873, over one hundred years ago. While the *Treatise* was not Maxwell's first expression of a full electromagnetic theory, it has had the greatest influence on physics since that time. It seems appropriate to review concisely Maxwell's work in electromagnetic theory. This discussion is not a full commentary on that work, which would require far more space than is available, but it does trace the development of electromagnetic theory and point out the principal paths which led Maxwell to the theory.

My approach is chronological, working through Maxwell's papers on electromagnetic theory and the *Treatise*. Four major papers must be considered:

1856 – "On Faraday's Lines of Force" [9].

1861-1862 – "On Physical Lines of Force" [10]-[13].

1864 – "A Dynamical Theory of the Electromagnetic Field" [14].

1868 – "A Note on the Electromagnetic Theory of Light" [15].

In addition, as we will see, one additional paper was never published [15]; and much correspondence [17]-[19] also sheds light on Maxwell's approach to electromagnetic theory, particularly in its early stages.

1 Maxwell's Early Work

James Clerk Maxwell graduated from Cambridge University in 1854. The first evidence we have of his interest in electrical situations occurred in a letter to a friend, William Thomson (later Lord Kelvin) in February of 1854 [20]:

> Now that I have entered the unholy estate of bachelorhood I have begun to think of reading. This is very pleasant for some time among books of acknowledged merit wh one has not read but ought to. But we have a strong tendency to return to Physical Subjects and several of us here wish to attack Electricity.

[1]Deceased, Dec. 18, 2007.

> Suppose a man to have a popular knowledge of electrical show experiments and a little antipathy to Murphy's Electricity [21], how ought he to proceed in reading & working so as to get a little insight into the subject wh may be of use in further reading?
> If he wish to read more Ampère Faraday &c how should they be arranged and at what state & in what order might he read your articles in the Cambridge Journal?
> If you have in your mind any answer to the above questions, three of us here would be content to look upon an embodiment of it in writing as advice.

Thomson's reply is not known, but he must have suggested a course of study for Maxwell in electromagnetic theory. Succeeding letters to Thomson report on the early progress of the budding electromagnetic theory and begin to show a simple theory developing in Maxwell's mind. Thus in November of 1854, Maxwell wrote [22]:

> I have heard very little of you for some time except thro' Hopkins & Stokes, but I suppose you are at work in Glasgow as usual. Do you remember a long letter you wrote me about electricity, for wh: I forget if I thanked you?
> I soon involved myself in that subject, thinking of every branch of it simultaneously, & have been rewarded of late by finding the whole mass of confusion beginning to clear up under the influence of a few simple ideas.
> As I wish to study the growth of ideas as well as the calculation of forces, and as I suspect from various statements of yours that you must have acquired your views by means of certain conceptions which I have found great help, I will set down for you the confessions of an electrical freshman.
> ⋮
> Now I have heard you speak of 'magnetic lines of force' & Faraday seems to make great use of them, but others seem to prefer the notion of attractions of elements of currents directly.

By September 1855 [23], there was a good bit more in the way of mathematical detail, and also more definite ideas about the direction in which such a theory might develop.

> I have got a good deal out of you on electrical subjects, both directly & through the printer & publishers & I have also used other helps, and read Faraday's three volumes of researches [24]-[26]. My object in doing so was of course to learn what had been done in electrical science, mathematical & experimental, and to try to comprehend the same in a rational manner by the aid of any notions I could screw into my head. In searching for these notions I have come upon some ready made, which I have appropriated. Of these are Faraday's theory of polarity which ascribes that property to every portion of the whole sphere of action of the magnetic or electric bodies, also his general notions about 'lines of force' with the 'conducting power' of different media for them.
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> I have also been working at Weber's theory of Electro Magnetism [27] as a mathematical speculation which I do not believe but which ought to be compared with others and certainly gives many true results at the expense of several startling assumptions.

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> I intend next to apply to these facts Faraday's notion of an *electrotonic* state. I have worked a good deal of mathematical material out of this vein and I believe I have got hold of several truths which will find a mathematical expression in the electrotonic state.

This mention of the electrotonic state is important.[2]

The concept, an idea of Faraday not at all well-known today, was to influence Maxwell greatly in his development of electromagnetic theory. The notion of the electrotonic state is important in all of Maxwell's work in electromagnetic theory, although both the name and the notation change from paper to paper. It is, as we will see, the origin of our present concept of the *vector potential*.

Faraday's notion came from thinking about the phenomenon of induction which he had studied experimentally. The electromotive force induced depends on the time rate of change of the magnetic flux; the flux is defined in terms of the whole region inside the coil of wire. Faraday felt somewhat unhappy about this, thinking that something happening directly at the position of the wire should lead to the EMF induced in the wire.[3] He was, therefore, led to postulate a new state of matter, the electrotonic state, that would account for the phenomenon of electromagnetic induction. The electrotonic state at the wire itself would thus lead to the EMF. Faraday's views about the electrotonic state were somewhat ambiguous. He supported the idea sometimes, but at other times thought there was little to it. But Maxwell was impressed with Faraday's concept, and, as we will see, gave it a mathematical form.

2 Faraday's Lines of Force

The first paper Maxwell published on electromagnetic theory was "On Faraday's Lines of Force" [9], published while he was a Fellow at Trinity College in Cambridge. We can trace this paper in early detail in his letters to Thomson just mentioned.

It begins on what modern physicists might regard as a serious note, on philosophical and methodological detail. Maxwell was clearly striving to grab hold somewhere in the bulk of confusing electro-magnetic material, but his hold was still precarious. Hence he considered how to develop a large and complicated theory such as a theory of electricity and magnetism, where the experimental and theoretical information both are in a confusing state. We can see this in the early passages of this paper.

> The present state of electrical science seems peculiarly unfavourable to speculation. ... No electrical theory can now be put forth, unless it shews the connexion not only between electricity at rest and current electricity, but between the attractions and inductive effects of electricity in both states. Such a theory must accurately satisfy those laws, the mathematical form of which is known, and must afford the means of calculating the effects in the limiting cases where the known formulae are inapplicable. In order therefore to appre-

[2] The concept of the *electro-tonic* state was first mentioned by Faraday in Art. 60 of [28]. Other mentions by Faraday include Art. 1661 of [29], Arts. 1729 and 1733 of [30], and Art. 3269 of [31].

[3] See Art. 1729 of [30]. This view is often invoked in discussions of the Aharonov-Bohm effect [32].

> ciate the requirements of the science, the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with progress. The first process therefore in the effectual study of the science, must be one of simplification and reduction of the results of previous investigation to a form in which the mind can grasp them. ... We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis.
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> In order to obtain physical ideas without adopting a physical theory we must make ourselves familiar with the existence of physical analogies. By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other.
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> This analogy between the formulae of heat and attraction was, I believe, first pointed out by Professor William Thomson in the Cambridge Math. Journal, Vol. III [35].
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> It is by the use of analogies of this kind that I have attempted to bring before the mind, in a convenient and manageable form, those mathematical ideas which are necessary to the study of the phenomena electricity. The methods are generally those suggested by the processes of reasoning which are found in the researches of Faraday, and which, though they have been interpreted mathematically by Prof. Thomson and others, are very generally supposed to be of an indefinite and unmathematical character, when compared with those employed by the professed mathematicians. By the method which I adopt, I hope to render it evident that I am not attempting to establish any physical theory of a science in which I have hardly made a single experiment, and that the limit of my design is to shew how, by a strict application of the ideas and methods of Faraday, the connexion of the very different orders of phenomena which he has discovered may be clearly placed before the mathematical mind.

Thus the process of physical analogy was going to guide Maxwell's thought, analogies from other areas of physics. Primarily the analogies he was employing are those from hydrodynamics and heat flow.

This paper does not present what we would currently regard as full electromagnetic theory, but the modern reader can begin to see, in component notation, forms of mathematical expression that would be familiar to him, those involving, for example, the vector calculus operators. The equations themselves were not new to physics but came through hydrodynamics; they are the same equations that we can find in Thomson's papers of this period. The experimental ideas present are mostly those of Faraday; in attempting to give mathematical form to such ideas as the electrotonic state, a major topic of this paper, Maxwell worked primarily within the conceptual framework set up by Faraday. There is no concept of displacement current in this first paper. The equations which were developed here appear, sometimes in modified form, in Maxwell's other work on electromagnetic theory.[4]

3 On Physical Lines of Force

Maxwell's second paper on electromagnetic theory was published in 1861 and 1862, in the Philosophical Magazine [10]-[13]. It has some claim to be considered as the "moment of discovery" of a full electromagnetic theory. It appears likely that Maxwell developed the missing concepts from his earlier paper in the process of writing this paper.

Parts 1 and 2 of "On Physical Lines of Force" are in the tradition of the earlier paper, although new notation (still a component notation) is introduced for all of the electromagnetic quantities. But the fundamental new ideas, and the notion of the electromagnetic theory of light, are absent from the first two parts of the paper. Indeed, these parts look self-contained; they end with a summary, as if Maxwell intended initially to end the paper at this point.

The paper is based on an extremely elaborate mechanical model, again in the spirit of a physical analogy such as Maxwell discussed in "On Faraday's Lines of Force". The model is much more complicated in some ways than anyone considered earlier, and certainly more mechanical in detail. Maxwell was generally clear about the status of this model; he did not consider it an actual representation of the world, but only as a model. Figure 1 is his diagram of that model.

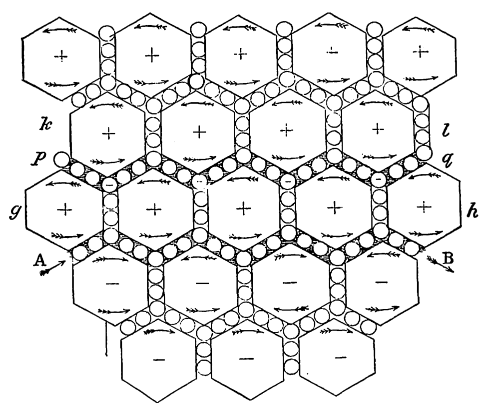

Figure 1: Maxwell's 1861 mechanical model of the electromagnetic field. From [11].

[4]Forms of all four of what are now called "Maxwell's equations" were already present in [9].

Maxwell discussed an integral form of Faraday's law on p. 50 of [9]: "The electro-motive force depends on the *change* in the number of lines of inductive magnetic action which pass through the circuit."

In eq. (C), p. 54, he stated that $\mathbf{\nabla} \cdot \mathbf{E} = 4\pi\rho$, with $\mathbf{E} = (a, b, c)$.

On p. 55, Maxwell introduced two magnetic fields, the magnetic "intensity" $\mathbf{H} = (\alpha_1, \beta_1, \gamma_1)$ and the magnetic "induction" $\mathbf{B} = (a_1, b_1, c_1)$ that obey his eq. (B), p. 53 in the form $\mathbf{H} = \mathbf{B}/\mu$, and eq. (C) in the form $\mathbf{\nabla} \cdot \mathbf{B} = 4\pi\rho_{\text{magnetic}}$. Maxwell did not state that $\rho_{\text{magnetic}} = 0$, although this was the view of Ampère [33]. On p. 56, Maxwell stated "Ampère's Law" in the form $\mathbf{J}_{\text{conduction}} = \mathbf{\nabla} \times \mathbf{H}$, with $\mathbf{J}_{\text{conduction}} = (a_2, b_2, c_2)$, after a brief mention of Ampère on p. 55.

Perhaps the best description of how the model is related to electric and magnetic quantities is that provided by Helmholtz [36]:

> ... a system of cells with elastic walls and cylindrical cavities ... in which elastic balls can rotate and be flattened out by the centrifugal force. In the walls of cells there must be other balls, of invariable volume, as friction rollers ... their center of gravity ... would merely be displaced by elastic yield of the cell-wall ... displacement of [the friction rollers] gives dielectric polarization of the medium; streaming of the same, an electric current; rotation of the elastic balls corresponds to the magnetizing of the medium, the axis of rotation being the direction of the magnetic force.

All of space is to be considered as full of such fluid cells. The rotation of the fluid corresponds to the magnetic field, and the volume of each of the cells corresponds to the electric field. The "idle wheels," the small ball bearings, are present because Maxwell realized that two successive vortices might be rotating in the same direction, because the magnetic field may be in the same direction in neighboring regions of space; so the little fluid cells would be moving in opposite directions at the point of contact. He followed the standard mechanical way of overcoming this by adding the idle wheels. This addition seems very contrived at first, but it turns out to be the real key to Maxwell's success with this model.

As indicated, the first two parts of the paper relate the model to the mathematical developments of the earlier paper. In general, with the entire paper, no deduction in the usual mathematical sense is present; rather the equations follow from the model itself.

In part 3, suddenly a very different spirit is present. The change is immediately dramatic. Maxwell began with a summary of previous work in this paper, focusing on the model itself. He said (pp. 12-13 of [12]):

> In the first part of this paper I have shown how the forces acting between magnets, electric currents, and matter capable of magnetic induction may be accounted for on the hypothesis of the magnetic field being occupied with innumerable vortices of revolving matter, their axes coinciding with the direction of the magnetic force at every point of the field.
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> I conceived the rotating matter to be the substance of certain cells, divided from each other by cell walls composed of particles which are very small compared with the cells, and that it is by the motions of these particles, and their tangential action on the substance in the cells, that the rotation is communicated from one cell to another.
>
> I have not attempted to explain this tangential action, but it is necessary to suppose, in order to account for the transmission of rotation from the exterior to the interior parts of each cell, that the substance in the cell possesses elasticity of figure, similar in kind, though different in degree, to that observed in solid bodies. The undulatory theory of light requires us to admit this kind of elasticity in the luminiferous medium, in order to account for transverse vibrations. We need not then be surprised if the magneto-electric medium possesses the same property.

We see that Maxwell was starting to talk about the theory of light, an idea which was not present in the first two parts of the paper. A page or so further along in the paper (p. 14 of [12]), he introduced a second new idea, the idea of displacement current:[5]

> In a dielectric under induction, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively, and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another.
> The effect of this action on the whole dielectric mass is to produce a general displacement of the electricity in a certain direction. This displacement does not amount to a current, because when it has attained a certain value it remains constant, but it is the commencement of a current, and its variations constitute currents in the positive or negative direction, according as the displacement is increasing or diminishing.

This quickly leads to the first statement one finds in Maxwell of the electromagnetic theory of light (p. 15 of [12]), again simply as a summary of what is to follow in more detail in the paper itself:

> I have deduced from this result the relation between the statica1 and dynamical measures of electricity, and have shown, by a comparison of the electromagnetic experiments of MM. Kohlrausch and Weber [37] with the velocity of light as found by M. Fizeau [38], that the elasticity of the magnetic medium in air is the same as that of the luminiferous medium, if these two coexistent, coextensive, elastic media are rather not one medium.

A little bit further in the paper (p. 15 of [12]), he made the first famous comparison between the value of the velocity of light, as derived from electromagnetic considerations, 193,088 miles per second, and the velocity of light as determined experimentally by Fizeau, 195,647. (A colleague of mine, William Parker,[6] has pointed out that these values of velocity are much closer to each other than they are to our currently accepted value.) Maxwell drew the obvious conclusion at this point.

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[5]In Maxwell's model, all space was a kind of dielectric medium, characterized by the displacement field $\mathbf{D} = (f, g, h)$, first introduced in [12]. This field was related to Maxwell's *electromotive force*, our electric field $\mathbf{E} = (P, Q, R)$, by the *elasticity* E^2 of the medium, *i.e.*, $\mathbf{D} = \epsilon\mathbf{E}$ where our dielectric constant ϵ is Maxwell's $1/4\pi E^2$. See eq. (105), p. 18 of [12], which equation (and several others in this paper) includes a minus sign that we now consider to be spurious.

Electric-charge density ρ (Maxwell's e in [12]) was associated with a strain in this medium according to $\rho = \boldsymbol{\nabla}\cdot\mathbf{D}$ in eq. (115), p. 19 of [12], and the *displacement current* (density) was $\partial\mathbf{D}/\partial t$ (p. 14 of [12]).

In our present view, a medium with dielectric polarization density $\mathbf{P}$ is associated with a bound electric-charge density $-\boldsymbol{\nabla}\cdot\mathbf{P}$ and a bound electric-current density $\partial\mathbf{P}/\partial t$. We write $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ (in SI units), and now consider that the term $\epsilon_0\partial\mathbf{E}/\partial t$ in the displacement-current density is not a physical electric-current density.

It remains that Maxwell's extension of "Ampère's law", his eq. (112), p. 19 of [12], is valid with its inclusion the "displacement current", although his model that inspired this insight is largely forgotten.

Maxwell also stated the other three of his "equations" in the 1861-62 papers. Faraday's law is given in differential form in eq. (54), p. 290 of [11]. The relation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ is given in eq. (56), p. 291 of [11] and again in eq. (72), p. 342 of [11]. The relation $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho_{\rm free}$ is given in eq. (115), p. 19 of [12].

[6]William Parker was Chairman of the Physics Department at Reed College when Bork was Professor there in the 1960's.

> The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena*.

The italics are Maxwell's!

It should be stressed that electromagnetic waves are not derived, in the sense of following as a logical conclusion from the fundamental laws of a mathematical theory. That is, no wave equation is derived from basic electromagnetic laws.[7] Instead everything follows on the basis of the model.

Maxwell's real excitement about the situation is reflected not only in the paper itself, but in a number of letters that he wrote at this time. The most detailed of these letters is probably the one to Michael Faraday on October 19, 1861 [39]. Maxwell appears to have had little direct contact with Faraday, being familiar with the Faraday work through published documents. So the letter is formal, rather than friendly. The relevant passages follow:

> Dear Sir — I have been lately studying the theory of static electric induction, and have endeavoured to form a mechanical conception of the part played by the particles of air, glass, or other dielectric in the electric field, the final result of which is the attraction and repulsion of 'charged' bodies.
>
> The conception I have hit on has led, when worked out mathematically, to some very interesting results, capable of testing my theory, and exhibiting numerical relations between optical, electric, and electromagnetic phenomena, which I hope soon to verify more completely.
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> I suppose the elasticity of the sphere to react on the electrical matter surrounding it, and press it downwards.
>
> From the determination by Kohlrausch and Weber of the numerical relation between the statical and magnetic effects of electricity, I have determined the elasticity of the medium in air, and assuming that it is the same with the luminiferous ether, I have determined the velocity of propagation of transverse vibrations.
>
> The result is 193,088 miles per second (deduced from electrical and magnetic experiments). Fizeau has determined the velocity of light = 193,118 miles per second, by direct experiment.
>
> This coincidence is not merely numerical. I worked out the formulae in the country before seeing Weber's number, which is in millimetres, and I think we have now strong reason to believe, whether my theory is a fact or not, that the luminiferous and the electromagnetic medium are one.
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[7] On p 22 of [12], Maxwell considered that electromagnetic "vibrations" propagate in an elastic medium, and the speed of light is related to the elasticity and the density of that medium, analogously to the speed of sound. His mechanical model provided expressions for these quantities, which led in his eq. (136) to the conclusion that the speed of the vibrations is very close to that of light.

> When I began to study electricity mathematically I avoided all the old traditions about forces acting at a distance, and after reading your papers as a first step to right thinking, I read the others, interpreting as I went on, but never allowing myself to explain anything by these forces. It is because I put off reading about electricity till I could do it without prejudice that I think I have been able to get hold of some of your ideas, such as the electrotonic state, action of contiguous parts, &c and my chief object in writing to you is to ascertain if I have got the same ideas which led you to see your way into things, or whether I have no right to call my notions by your names.

It is interesting to note that the two values quoted here are even closer together because the measurement of Fizeau is different than in the paper.

Maxwell also wrote to Thomson on 10 December 1861 [40], and reported the results informally, including a number of values for the velocity of light measurement, perhaps explaining the discrepancy between the two earlier results:

> Since I saw you I have been trying to develop the dynamical theory of magnetism as an affection of the whole magnetic field according to the views stated by you. ...
>
> I have calculated the relation between the force and the displacement on the supposition that the cells are spherical and that their cubic and linear elasticities are connected as in a 'perfect' solid. I have found from this the attraction between two bodies having given quantities of free electricity on their surfaces, and then by comparison with Weber's value of the statical measure of a unit of electrical current I have deduced the relation between the elasticity and density of the cells. The velocity of transverse undulations follows from this directly and is equal to 193,088 miles per second, very nearly that of light.

Velocity of light miles per sec.	192,500 by abberation 195,777 by Fizeau 193,118 Galbraith and Haughton's statement [41] of Fizeau's results

> I made out the equations in the country before I had any suspicion of the measures between the two values of the velocity of propagation of magnetic effects and that of light, so I think I have reason to believe that the magnetic and luminiferous media are identical and that Weber's number is really, as it appears to be, one half of the velocity of light in millimeters per second.

The emphasis of having done it in the country indicates that he was anxious to show that these figures were not rigged, but represented an honest comparison.

In spite of all this writing to friends, and the paper itself, there seems to have been little immediate interest in this result.

4 A Dynamical Theory of the Electromagnetic Field

In 1864 Maxwell published a major paper on the electromagnetic field in the Proceedings of the Royal Society [14]. The contrast in *content* between this paper and the last part of the

1861-1862 paper is inconsequential but the contrast in *style* of presentation is very great. While in the 1861-62 paper everything depends on the mathematical model, here almost no mention of this model occurs. In fact almost no reference is made to any previous Maxwell papers in electromagnetic theory. The 1864 paper is in the traditional style of a typical mathematical physics paper, axiomatic in structure and revealing almost nothing about how the various fundamental equations were obtained.

The theory in this paper is based on 20 equations in 20 unknowns, a number that may startle the modern reader. Figure 2 presents Maxwell's basic equations, both in the component notation he is using, and also "translated" into a contemporary vector notation. Such translation has its dangers, but it is useful in comparing Maxwell's theory with contemporary presentations.

In the equations of the electromagnetic field we have assumed twenty variable quantities, namely:

Quantity	Maxwell's Notation	Modern Notation
Electromagnetic Momentum	$F,\ G,\ H$	$\mathbf{A}$
Magnetic Intensity	$\alpha,\ \beta,\ \gamma,$	$\mathbf{H} = \mathbf{B}/\mu$
Electromotive Force	$P,\ Q,\ R$	$\mathbf{E}$
Current due to true conduction	$p,\ q,\ r$	$\mathbf{J}$
Electric Displacement	$f,\ g,\ h$	$\mathbf{D}$
Total Current (including variation of displacement)	$p',\ q',\ r'$	$\mathbf{J}_T = \mathbf{J} + \partial\mathbf{D}/\partial t$
Quantity of free Electricity	e	ρ
Electric Potential	Ψ	ϕ or V

Between these twenty quantities we have found twenty (scalar) equations, *viz.*:

Quantity	Maxwell's Notation	Modern Notation
Total Currents (A)	$(p',q',r') = (p,q,r) + \frac{\partial(f,g,h)}{\partial t}$	$\mathbf{J}_T = \mathbf{J} + \partial\mathbf{D}/\partial t$
Magnetic Force (B)	$\mu\alpha = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}$, *etc.*	$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$
Electric Currents (C)	$\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z} = 4\pi p'$, *etc.*	$\boldsymbol{\nabla}\times\mathbf{H} = 4\pi\mathbf{J}_T$
Electromotive Force (D)	$P = \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{\partial F}{\partial t} - \frac{\partial\Psi}{\partial x}$, *etc.*	$\mathbf{F}/q = \mathbf{v}\times\mathbf{B} - \frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}V$
Electric Elasticity (E)	$(p,q,r) = k(f,g,h)$	$\mathbf{E} = \mathbf{D}/\epsilon$
Electric Resistance (F)	$(P,Q,R) = \varrho(p,q,r)$	$\mathbf{E} = \mathbf{J}/\sigma$
Free Electricity (G)	$e + \partial f/\partial x + \partial g/\partial y + \partial h/\partial z = 0$	$\rho + \boldsymbol{\nabla}\cdot\mathbf{D} = 0$
Continuity (H)	$\partial e/\partial t + \partial p/\partial x + \partial q/\partial y + \partial r/\partial z = 0$	$\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$

Figure 2: Summary of Maxwell's basic equations, from secs. 53-70 of [14]. Note the sign error in Maxwell's equation (G).

The modern reader will clearly recognize many of these equations, particularly when they are translated to the contemporary notation. Some of Maxwell's names are still different from our current names. For example, in this paper the quantity whose components are F, G, and H, earlier called the components of the electrotonic state, is called the *electromagnetic momentum*. The reason for this choice of name is not hard to understand, if we consider the relation between the electric field, which is (electrostatically) a force on a unit charge and the electromagnetic momentum. If we neglect the scalar potential (*i.e.*, the term $-\boldsymbol{\nabla}\Psi$), this relation in modern notation will look as follows:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}\,.$$

The name comes from the comparison between this and the usual form of writing Newton's second Law in terms of momentum,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}\,.$$

Perhaps the most striking immediate difference between Maxwell's formulation and the typical contemporary presentation of "Maxwell's equations" is that the equations connecting the fields and the potentials are considered as part of the fundamental set, rather than as auxiliary mathematical aids. Maxwell did not share the often expressed attitude that the potentials are less "important" or "real" than the fields. The equation (D) relating $\mathbf{E} = (P, Q, R)$ to the potentials also has an extra term, related in Maxwell's mind to the medium in which the fields exist; the structure of this term indicates that we have "moved" it to the Lorentz force equation.

In "A Dynamical Theory" [14] the wave equation is, for the first time, *derived* from electromagnetic equations. The following logical flow chart (Fig. 3), reprinted from a paper [5] in the American Journal of Physics, shows Maxwell's argument; the lines and arrows indicate which results are used to demonstrate what.

Again the notation is translated to a present-day form. We see that Maxwell derived the wave equation for the electromagnetic momentum, "our" vector potential. His attitude toward the gauge condition he used ($\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ in our notation) was an important consideration in this derivation.

A letter (5 January 1865 [42]) to C.H. Cay,[8] a personal friend, shows Maxwell's excitement:

> I have also a paper afloat, with an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns.

5 A Note on the Electromagnetic Theory of Light

Just as with the 1861-1862 paper [10]-[13], the more formal 1864 paper [14] appeared to fall on deaf ears. Even though Maxwell was already, because of other work, the leading figure in British physics, these papers were ignored even by such close friends as William Thomson and Peter Tait. Meanwhile the German physicists were continuing to publish electromagnetic results using the "action-at-a-distance" view of mechanics.

The "Note on The Electromagnetic Theory of Light" was published in 1868 as part of a much larger and more detailed paper [15] on electromagnetic measurements. The formulation

[8]Maxwell's mother's maiden name was Cay. Regarding Maxwell's personal and scientific relations with the Cay family, see, for example, [43].

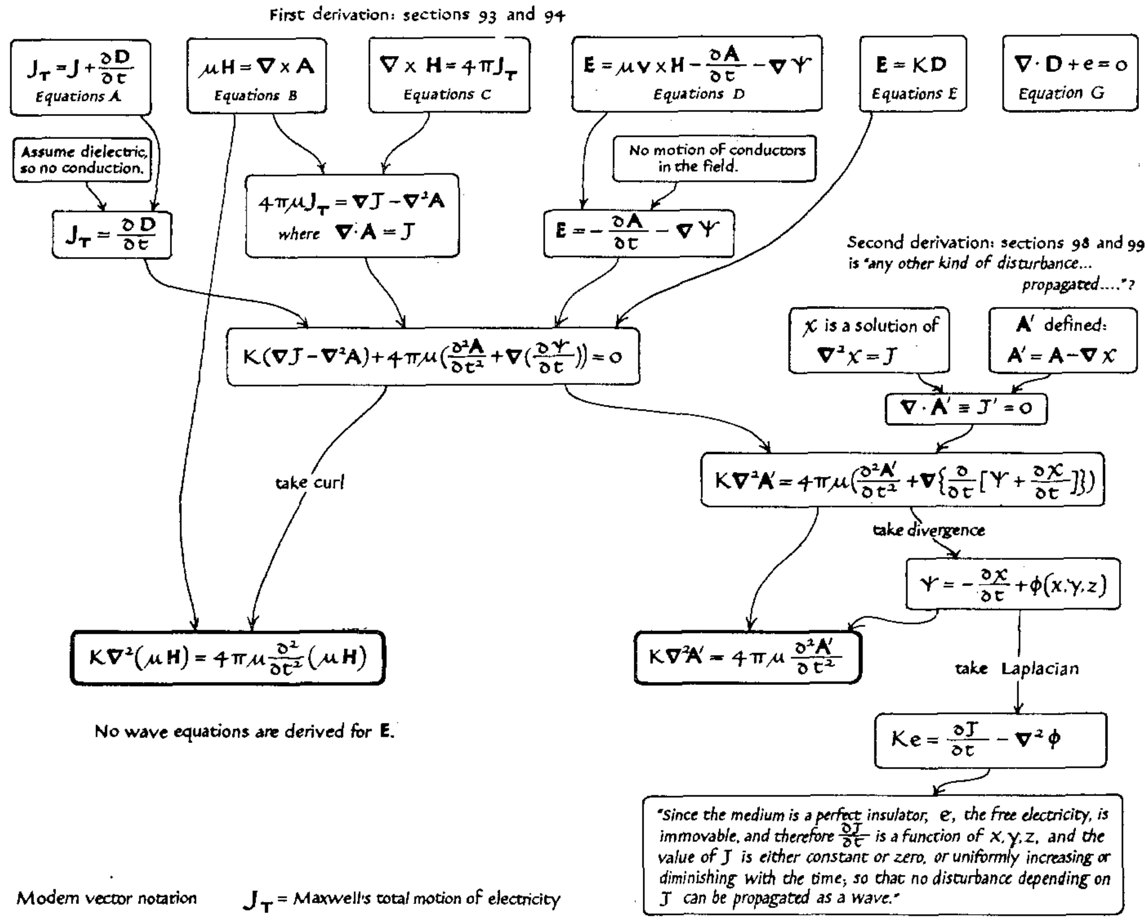

Figure 3: Flow chart of Maxwell's derivation of the electromagnetic wave equations in [14].

of electromagnetic theory in the short "Note" is unlike that found anywhere else in the Maxwell canon; it is a verbally stated *integral* formulation of the basic equations, one that Maxwell was not to repeat later in the *Treatise on Electricity and Magnetism.* Four theorems are stated, corresponding closely to our present-day formulation of the theory in terms of integral equations. Again, the wave equation is deduced, but here restricted immediately to a plane wave moving along one of the coordinate axes. Maxwell was trying to present the briefest and clearest possible demonstration of the result, and so was abandoning a good bit of the mechanism of the 1864 paper. We might interpret his not following this procedure in the *Treatise* as an indication that it led to no greater degree of success in gathering adherence to the electromagnetic point of view than did the earlier papers.

The following chart (Fig. 4), also reprinted from the American Journal of Physics [5], sketches Maxwell's derivation of the electromagnetic wave equation in [15].

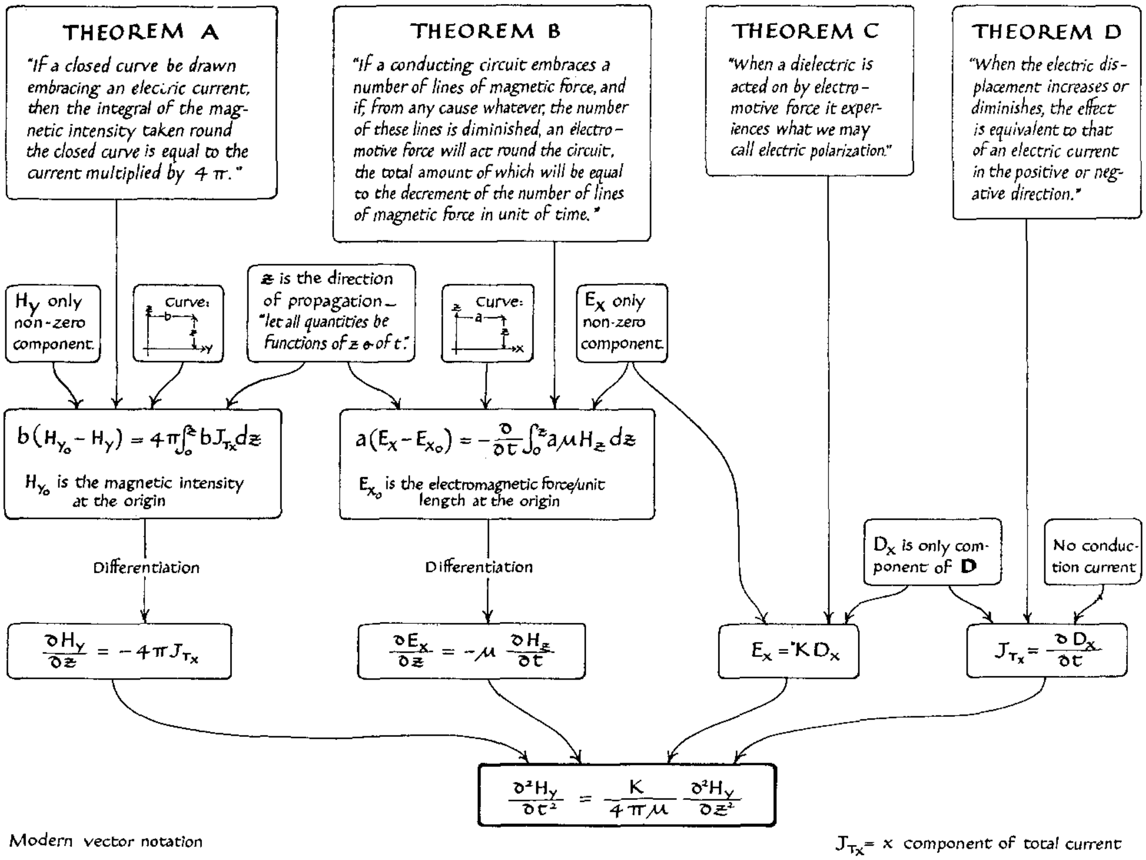

Figure 4: Flow chart of Maxwell's derivation of the electromagnetic wave equation in [15].

6 Maxwell's Rejected Paper

Maxwell also submitted another paper in 1864 to the Royal Society [16]. But this paper was rejected and was never printed. It does exist in manuscript form in the Maxwell manuscripts in the University Library at Cambridge, however, and the letter from G.G. Stokes rejecting it (also in Cambridge) has been reprinted although without knowledge of the Maxwell paper.

This paper is very different, however, from any of the other papers we are examining, and very different from the Maxwell tradition generally. It is a report of an *experiment* to determine the velocity of light through the luminiferous ether!

It is not well known that Maxwell was highly interested in this problem, and pursued it in small ways from this time, 1864, until he died about fifteen years later. Maxwell's interest led directly to Michelson's initial experiment. Maxwell's too is a null result, but a very tentative one. Stokes questioned the basis of it, particularly with regard to some assumptions about refraction that were made in the paper. Experimental details are very different from the ones that Michelson first and then Michelson and Morley later were to pursue.

The other references to Maxwell's interest in this problem are also obscure. They occurred in letters to others, and thus appeared in papers and letters of others. Thus the British astronomer William Huggins [44] reprinted material from a letter of Maxwell's [45] inquiring about astronomical possibilities for making this type of measurement.

7 The *Treatise on Electricity and Magnetism*

Undoubtedly the most famous Maxwell work on electromagnetic theory and the one fundamental to almost all the later developers of the theory was the *Treatise on Electricity and Magnetism*, first published in 1873 [7, 8]. This is as indicated a formal two-volume treatise, and so has much more detail of electromagnetic phenomena and theory than is present in the Maxwell papers. It was an attempt to put the whole structure of electromagnetic theory in a somewhat logical order.

Maxwell's *Treatise* determined the structure of most of the treatises and textbooks in electromagnetic theory since his time; it begins first with electrostatics, and then moves successively through different areas of electric and magnetic theory before presenting the full theory. Maxwell, it appears, never intended the *Treatise* to be a text, so it is somewhat ironic that the structure of this book has so influenced physics texts in this area in the last 100 years.

The basic equations of electromagnetic theory, and the derivations based on these equations, are very similar to those in the 1864 paper [14]; it is fair to characterize most of the differences in fundamental theory as minor. In terms of the consequences produced from the theory, little is here that was not present in the 1864 paper. One new physical case handled is crystal optics, a problem not considered in the earlier papers. Again Maxwell deduced electromagnetic waves from the fundamental equations of electromagnetic theory, here in two independent deductions. The derivations differ in detail from those in the 1864 paper, but they are both similar in spirit. The following chart (Fig. 5), also reprinted from the American Journal of Physics [5], gives the logical structure of these two derivations.

Perhaps the principal innovation in general electromagnetic theory found in the *Treatise* is the introduction of a new notational scheme in places, the quaternions. As we have seen, although Maxwell used the term vector in his papers, and spoke of the components of a vector, he used a notation where the components of a vector were indicated by three successive letters. A few years before the *Treatise*, his correspondence to Tait suddenly moved to the question of quaternions. Peter Tait considered himself to be the Hamiltonian descendent of the quaternion tradition, and saw, as did Hamilton, the quaternion as being a key mathematical idea in all parts of physics. Tait and Maxwell had been school friends, and maintained a lifelong friendship and correspondence. Tait saw that the kinds of things Maxwell was doing in electromagnetic theory were related to quaternions, and made a serious effort in this correspondence to persuade him to use the quaternion formalism. We cannot follow all the details of this correspondence, but a few excerpts from letters may be useful in giving the spirit of it:

Maxwell, March 7, 1865 [46]

> Does anyone write quaternions but Sir W. Hamilton and you? I heard him greatly slanged by a mathematical clergyman unknown to me, along with Plucker and Jacobi. We were to see an end of all that school of mathematics very soon.

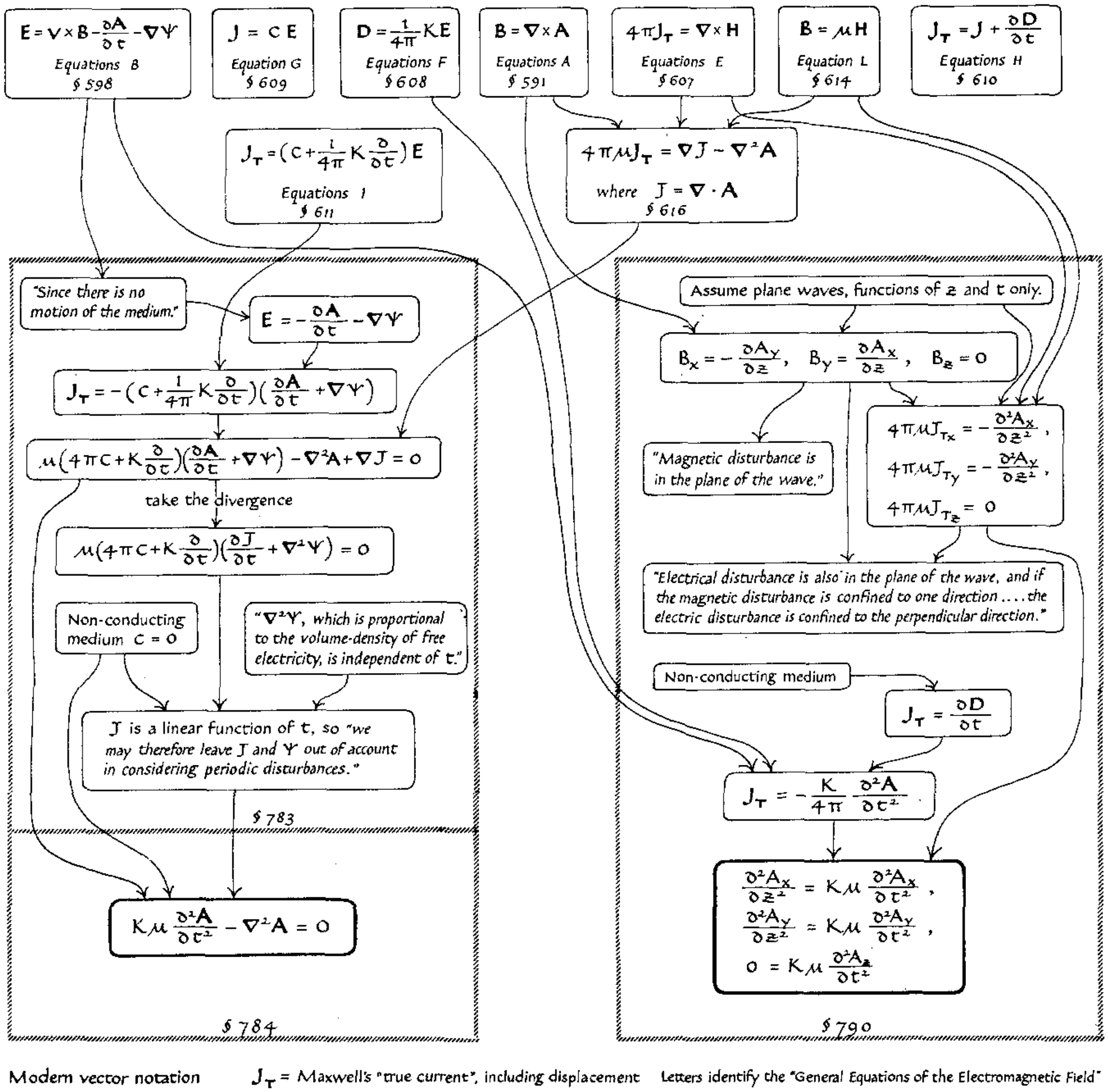

Figure 5: Flow chart of Maxwell's derivation of the electromagnetic wave equations in [8].

Tait, December 13, 1867

> If you read the last 20 or 30 pages of my book [47] I think you will see that 4^{nions} are worth getting up, for there it is shown that they go into that Δ business like greased lightning. Unfortunately I cannot find time to work steadily at them.

Maxwell, November 7, 1870 [48] (see also [49])

> Dear Tait
>
> $$\boldsymbol{\nabla} = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$
>
> What do you call this? Atled?

I want to get a name or names for the result of it on scalar or vector functions of the vector of a point. Here are some rough hewn names. Will you, like a good Divinity shape their ends properly so as to make them stick.
(1) The result of ∇ applied to a scalar function might be called the slope of the function. Lamé would call it the differential parameter but the thing itself is a vector, now slope is a vector word, whereas parameter has, to say the least, a scalar sound.
(2) If the original function is a vector then ∇ applied to it may give two parts. The scalar part I would call the *Convergence* of the vector function and the vector part I would call the *twist* of the vector function. (Here the word *twist* has nothing to do with a screw or helix. If the word *turn* or *version* would do they would be better than twist for twist suggests a screw. *Twirl* is free from the screw notion and is sufficiently racy. Perhaps it is too dynamical for pure mathematics so for Cayley's sake I might say *curl* (after the fashion of Scroll).
⋮
What I want to ascertain from you is if there are any better names for these things, or if these names are inconsistent with anything in Quaternions, for I am unlearned in quaternion idioms and may make solecisms.
I want phrases of this kind to make statements in electromagnetism and I do not wish to expose either myself to the contempt of the initiated, or Quaternions to the scorn of the profane.

Maxwell, November 14, 1870 [50]

> With regard to my dabbling in Hamilton I want to leaven my book with Hamiltonian ideas without casting the operations into a Hamiltonian form for which neither I, nor, I think, the public are ripe. Now the value of Hamilton's idea of a Vector is unspeakable and so are those of the addition and multiplication of vectors I consider the form into which he put these ideas, such as the names of Tensor Verser, Quaternion etc important and useful but subject to the approval of the mathematical world.

Maxwell, November 2, 1871 [51]

> The unbelievers are rampant. They say 'show me something done by 4^{nions} which has not been done by old plans. At best it must rank with abbreviated notation'.
> You should reply to this, no doubt you will. But the virtue of the 4^{nions} lies not so much as yet in solving hard questions as in enabling us to see the meaning of the question and its solution.

Maxwell, October 4, 1872 [52]

> How about electromagnetic 4^{nions} as in proof slip 106, 107? which please annotate and return. I suspect I am not sufficiently free with the use of the Tensor symbol in devectorizing such things as r (the distance between two points). The great want of the lay is a grammar of 4^{nions} in the form of rules as to annotation & interpretation not only of S, T, U, V but of (both kinds) and the proper position of $d\sigma$ &c. Contents, Notation, Syntax, Prosody, Nablady.

Maxwell's final use of the quaternions in the *Treatise* was somewhat minor. They were seldom used in derivations; that is, he did not seem well enough at home with quaternion formalism to deduce anything using the mechanism provided. But he saw quaternions as valuable aids to understanding, in the sense that they furnish a concise way to write the important results. Thus in the only place in the *Treatise* where the fundamental electromagnetic equations are written as a group, quaternion notation is used. Maxwell departed from the Hamilton-Tait tradition by using German Gothic letters to indicate the quaternions. Figure 6 gives examples of this from the *Treatise*.

Quantity	Maxwell's Notation	Modern Notation
Electromagnetic momentum	$\mathfrak{A} = (F, G, H)$	$\mathbf{A}$
Magnetic induction	$\mathfrak{B} = (a, b, c)$	$\mathbf{B}$
Magnetic force	$\mathfrak{H} = (\alpha, \beta, \gamma)$,	$\mathbf{H}$
Electromotive force	$\mathfrak{E}$ or $\mathfrak{F} = (P, Q, R)$	$\mathbf{E}$ or $\mathbf{F}/q$
Velocity of a point	$\mathfrak{V} = (\dot{x}, \dot{y}, \dot{z})$	$\mathbf{v}$
Current of conduction	$\mathfrak{K} = p,\ q,\ r$	$\mathbf{J}$
(Total) electric current	$\mathfrak{J} = (u, v, w)$	$\mathbf{J}_T$
Electric displacement	$\mathfrak{D} = (f, g, h)$	$\mathbf{D}$
Intensity of Magnetization	$\mathfrak{I} = (A, B, C)$	$\mathbf{M}$
Electric density	e	ρ
Electric potential	Ψ	ϕ or V

Quantity	Maxwell's Notation	Modern Notation
Magnetic induction (A)	$\mathfrak{B} = V\nabla\mathfrak{A}$	$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$
Electromotive force (B)	$\mathfrak{F} = V\mathfrak{V}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\Psi$	$\mathbf{F}/q = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} V$
Magnetic force (density) (C)	$\mathfrak{f} = V\mathfrak{J}\mathfrak{B}$	$\mathbf{f} = \mathbf{J}_T \times \mathbf{B}$
Magnetization (D)	$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{I}$	$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$
Electric Currents (E)	$4\pi\mathbf{J} = V\nabla\mathfrak{H}$	$4\pi\mathbf{J}_T = \boldsymbol{\nabla} \times \mathbf{H}$
Electric displacement (F)	$\mathfrak{D} = \frac{K}{4\pi}\mathfrak{E}$	$\mathbf{D} = \epsilon\mathbf{E}$
Electric Resistance (G)	$\mathfrak{K} = C\mathfrak{E}$	$\mathbf{J} = \sigma\mathbf{E}$
True Currents (H)	$\mathfrak{J} = \mathfrak{K} + \dot{\mathfrak{D}}$	$\mathbf{J}_T = \mathbf{J} + \partial\mathbf{D}/\partial t$
Free Electricity (J)	$e = S\nabla\mathfrak{D}$	$\rho = \boldsymbol{\nabla} \cdot \mathbf{D}$
Magnetic induction and force (L)	$\mathfrak{B} = \mu\mathfrak{H}$	$\mathbf{B} = \mu\mathbf{H}$

Figure 6: Maxwell's equations in quaternion notation, from Arts. 591-619 of [8]. Maxwell did not clearly distinguish between the (Lorentz) force per unit charge, $\mathbf{F}/q$ when in motion, and the electric field $\mathbf{E} = -d\mathbf{A}/dt - \boldsymbol{\nabla}\Psi$.

Hamilton's del (∇) operator is in evidence, having the same meaning that we use today. The quaternions are a combination of a vector plus a scalar (an allowable combination), with only one kind of product, the quaternion product. The rules of quaternion algebra for the "unit" vectors i, j, and k are as follows:

$$i \times i = j \times j = k \times k = -1$$
$$i \times j = k = -j \times i, etc.$$

Quaternion algebra is associative and distributive, but not commutative. S and V indicate the scalar part of a quaternion and the vector part of a quaternion, respectively.

The German Gothic letters are the origin of our own notation for the electric and magnetic fields. Later writers, particularly Oliver Heaviside [53] and Willard Gibbs [54], were, in trying to use electromagnetic theory, to shy away from the full quaternion mechanism, and pull out a subset particularly adapted to physics, the vector calculus. Their main interest was in *developing* electromagnetic theory, and so the mathematical and philosophical issues which motivated Hamilton and Tait were not critical to the followers of Maxwell.

One minor contribution of the use of quaternions in the *Treatise on Electricity and Magnetism* was the name *vector potential*, the third name, as we have seen, that Maxwell used for this entity (the quaternion $\mathfrak{A}$). In (Art. 617 of) the *Treatise*, Maxwell noted that, in the gauge $S\nabla\mathfrak{A} = \boldsymbol{\nabla} \cdot \mathfrak{A} = 0$, the field $\mathfrak{A}$ obeys a form of Poisson's equation, as does the electrostatic potential. It can be seen then (using eqs. (A), (E) and (L) of Fig. 6) that the mathematical structure $4\pi\mu\mathfrak{H} = -V\nabla\mathfrak{A}$ is, in quaternion notation, exactly the same as that for the electrostatic field, $\mathfrak{E} = -\nabla\Psi$, and this justified in Maxwell's eyes the name "vector potential." The similarity is not as close with our present vector notation.

Another minor, but interesting, aspect of the *Treatise on Electricity and Magnetism* is the introduction by Maxwell of the terms "convergence" (the negative of divergence), "curl,"[9] and "gradient." These occur in an early section (Art. 25) in Volume 1 [7], but they were not actually used by Maxwell elsewhere in the *Treatise*. However, later readers, particularly Heaviside, took these terms seriously, and so they became common language.

8 Later Development in Electromagnetic Theory

Toward the end of Maxwell's life, younger physicists, particularly George Francis FitzGerald [56] and H.A. Lorentz [57], began not only to read Maxwell, but to pursue electromagnetic theory beyond where Maxwell had taken it. Perhaps the major succeeding stage in theoretical development occurred separately in the work of Heinrich Hertz [58] and Oliver Heaviside [60], both of whom independently rewrote the Maxwell theory so that the equations looked pretty much the way that we see them today: Heaviside used a vector notation, while Hertz still used a component notation. Heaviside was particularly anxious to remove the potentials from the fundamental theory, and it is his attitude toward potentials that is usually put in Maxwell's mouth in contemporary treatments.

The single event which did most to promote the case of Maxwell's electromagnetic theory was Hertz's experimental production of electromagnetic waves other than light waves, a well-documented occurrence [59]. After that time work on electromagnetic theory increased, and

[9]The term "curl" was changed to "rotation" by W.D. Niven in [55] after Maxwell's death.

by 1900 it was already an accepted theory in physics. It was the exciting new theory of the time, and so it is not surprising that younger physicists., such as Albert Einstein, were very much influenced by the new theory. But that is another story.

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