“Hidden” Momentum in an Isolated Brick?

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1 Problem

The term “hidden” momentum was popularized by Shockley [1] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

A definition of “hidden” momentum has been proposed [2] (see also [3]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\text{cm}})\,d\text{Vol}, \quad (1)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,, \quad (2)$$

is the 4-force density due to the subsystem, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

Consider an isolated (unstressed) brick in an inertial frame where it has constant velocity $v\,\hat{\mathbf{x}}$ parallel to, say, its longest dimension l. In this frame the brick has mass m. The total “hidden” momentum of this brick is zero, so consider a partition of the brick into two subsystems by an imaginary, moving surface, $x_{\text{boundary}} = a + ut$ for $0 < a < l$, while the brick extends over $vt < x < l + vt$. What is the “hidden” momentum in the portion of the brick at $0 < x < a$ at time $t = 0$?

2 Solution

Labeling the subsystem at $vt < x < a + ut$ by the subscript $_a$, its time-dependent mass is,

$$m_a = \frac{a + (u - v)t}{l} m. \quad (3)$$

The time-dependent momentum of this subsystem is,

$$\mathbf{P}_a = m_a\, v\, \hat{\mathbf{x}}. \quad (4)$$

The center of mass of the subsystem has x-coordinate (for times when the boundary surface is within the brick),

$$x_{\rm a,cm} = vt + \frac{a + (u - v)t}{2}\,, \tag{5}$$

and the velocity of the center of mass is,

$$\mathbf{v}_{\rm a,cm} = \frac{\mathbf{u} + \mathbf{v}}{2}\,. \tag{6}$$

Thus,

$$\mathbf{P}_a - m_a \mathbf{v}_{\rm a,cm} = m_a \left(\mathbf{v} - \frac{\mathbf{u} + \mathbf{v}}{2} \right) = \frac{ma(\mathbf{v} - \mathbf{u})}{2l}\,, \tag{7}$$

which is nonzero for unless $u = v$.

We now evaluate the "hidden" momentum at time $t = 0$ according to the first form of eq. (1). The boundaries in x of subsystem a at this time are at $x = 0$ and a, with velocities v and u, while $x_{\rm a,cm}(0) = a/2$:

$$\begin{aligned}
\mathbf{P}_{\rm a,hidden} &= \mathbf{P}_a - m_a \mathbf{v}_{\rm a,cm} + \left(0 - \frac{a}{2}\right) \frac{m}{l} (\mathbf{v} - \mathbf{v}) - \left(a - \frac{a}{2}\right) \frac{m}{l} (\mathbf{v} - \mathbf{u}) \\
&= \frac{ma}{2l}\,(\mathbf{v} - \mathbf{u}) - \frac{ma}{2l}\,(\mathbf{v} - \mathbf{u}) = 0.
\end{aligned} \tag{8}$$

To use the second form of eq. (1), we note that $T^{00} = mc^2/Al$ and $T^{0x} = mcv/Al$ are constant within subsystem a, whose cross-sectional area is A, while $T^{0y} = 0 = T^{0z}$ everywhere. The time component f^0 of the 4-force density (2) is,

$$f^0 = \partial_0 T^{00} + \partial_i T^{0i} = \frac{\partial T^{00}}{\partial ct} + \frac{\partial T^{0x}}{\partial x}\,. \tag{9}$$

The time dependence of T^{00} of subsystem a is due only to the presence of its moving boundaries, $x = vt$ and $x = a + ut$. Near the "left" boundary, $T^{00}(x,t) = T^{00}(x - vt)$, such that $\partial T^{00}/\partial ct = -(v/c)\partial T^{00}/\partial x$, while near the "right" boundary, $T^{00}(x,t) = T^{00}(x - a - ut)$, such that $\partial T^{00}/\partial ct = -(u/c)\partial T^{00}/\partial x$.

It suffices to complete the calculation of $\mathbf{P}_{\rm a,hidden}$ for $t = 0$, as the result should be independent of time. At $t = 0$, f^0 is nonzero only at/near the boundaries $x = 0$ and $x = a$, so we can split the integration to that over $0 < x < b < a$ and $b < x < a$,

$$\begin{aligned}
P_{{\rm hidden},x}(t = 0) &= -\int \frac{f^0}{c}(x - x_{\rm cm})\, d{\rm Vol} = -\frac{A}{c} \int_0^a dx\, f^0 \left(x - \frac{a}{2}\right) \\
= \frac{Av}{c^2} \int_0^b dx\, \frac{\partial T^{00}}{\partial x} \left(x - \frac{a}{2}\right) &+ \frac{Au}{c^2} \int_b^a dx\, \frac{\partial T^{00}}{\partial x} \left(x - \frac{a}{2}\right) - \frac{A}{c} \int_0^a dx\, \frac{\partial T^{0x}}{\partial x} \left(x - \frac{a}{2}\right) \\
&= \frac{Av}{c^2} \left[x T^{00} \big|_0^b - \int_0^b dx\, T^{00} - \frac{a}{2}[T^{00}(b) - T^{00}(0)] \right] \\
&\quad + \frac{Au}{c^2} \left[x T^{00} \big|_b^a - \int_b^a dx\, T^{00} - \frac{a}{2}[T^{00}(a) - T^{00}(b)] \right] \\
&\quad - \frac{A}{c} \left[x T^{0x} \big|_0^a - \int_0^a dx\, T^{0x} - \frac{a}{2}[T^{0x}(a) - T^{0x}(0)] \right] = 0,
\end{aligned} \tag{10}$$

in agreement with eq. (8), taking T^{00} and T^{0x} to have values their nonzero, constant values within the interval $0 \le x \le a$. and zero outside this.[1]

Thus, according to the calculations (9) and (10), there is no "hidden" momentum in the "all-mechanical" example of an isolated brick, or in subsystems of it defined by moving partitions.

As noted in sec. VI of [4], "hidden" momentum is associated with (sub)systems that have internal motion when "at rest", which is not the case for an isolated brick.

References

[1] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[2] D. Vanzella, Private communication, (June 29, 2012).

[3] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[4] D. Babson *et al.*, *Hidden momentum, field momentum, and electromagnetic impulse*, Am. J. Phys. **77**, 826 (2009), http://kirkmcd.princeton.edu/examples/EM/babson_ajp_77_826_09.pdf

[1] An analysis of eq. (10) which invokes Heaviside step functions Θ, and Dirac delta functions δ, notes that in the frame where the rod has velocity v, the nonzero components of $T^{0\mu}$ can be written as,

$$T^{00} = \frac{mc^2}{Al}[\Theta(x - vt) - \Theta(x - a - ut)], \tag{11}$$

$$T^{0x} = \frac{mcv}{Al}[\Theta(x - vt) - \Theta(x - a - ut)], \tag{12}$$

where $\Theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$. Then,

$$\frac{\partial T^{00}}{\partial ct} = -\frac{mc^2}{Al}\left[\frac{v}{c}\delta(x - vt) - \frac{u}{c}\delta(x - a - ut)\right], \tag{13}$$

$$\frac{\partial T^{0x}}{\partial x} = \frac{mcv}{Al}[\delta(x - vt) - \delta(x - a - ut)], \tag{14}$$

$$\begin{aligned} P_{\mathrm{hidden},x}(t=0) &= -\int \frac{f^0}{c}(x - x_{\mathrm{cm}})\,d\mathrm{Vol} = -\frac{A}{c}\int_0^a dx\left(\frac{\partial T(00}{\partial ct} + \frac{\partial T^{0x}}{\partial x}\right)(x - x_{\mathrm{cm}}) \\ &= \frac{A}{c}\int_0^a dx\,\frac{mc^2}{Al}\left[\frac{v}{c}\delta(x) - \frac{u}{c}\delta(x - a)\right](x - a/2) - \frac{A}{c}\int_0^a dx\,\frac{mcv}{Al}[\delta(x) - \delta(x - a)](x - a/2) \\ &= \frac{ma}{l}\left(-u - \frac{v}{2} + \frac{u}{2}\right) + \frac{mav}{l} = \frac{ma(v - u)}{2l}\,. \end{aligned} \tag{15}$$

Note that the result of eq. (15) is the same as $\mathbf{P}_a - m_a\mathbf{v}_{\mathrm{a,cm}}$ of eq. (7). Hence, if the boundary integral in the first form of eq. (1) were ignored, the two forms of that expression, according to calculations using delta functions in f^0, would both lead to the same, nonzero "hidden" momentum in present example.

This author finds the delta functions in the expressions (13)-(14) for the 4-force density f^μ very unappealing physically, and so prefers the analysis in the main text that avoids them, with the implication that there is zero "hidden" momentum in the present example.