Leaky Bucket Suspended by a Spring

Johann Otto
Höhere Technische Bundeslehr- und Versuchsanstalt, Wiener Neustadt, A-2700 Austria
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(May 7, 2018; updated May 25, 2018)

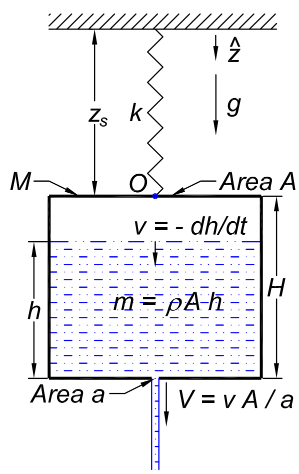
1 Problem

Discuss the vertical motion of a leaky bucket of water that is suspended by a spring.

This is an extension of the classic example of Torricelli [1]-[4] of water emerging from a hole in a water tank.

2 Solution

We take the bucket to be a right circular cylinder of mass M and base area A and height H, whose point of suspension is at distance z_s below the upper, fixed point of a spring of constant k and (for simplicity) zero rest length. Initially, the bucket is filled to height h_0 with (incompressible, inviscid) water of constant mass density ρ and negligible viscosity. A circular hole of area a, not necessarily small compared to area A, exists in the center of the base of the bucket.



The variable mass of the bucket plus the water still inside it is,

$$M_{\rm tot}(t) = M + \rho A h(t), \tag{1}$$

when the water level above the base of the bucket is $h(t)$ at time t.

The velocity of the water level in the bucket is, in the convention that the z-axis is positive downwards,

$$\mathbf{v} = -\frac{dh}{dt}\,\hat{\mathbf{z}} \equiv -\dot{h}\,\hat{\mathbf{z}} \qquad (v = -\dot{h}). \tag{2}$$

and the velocity of the efflux (water leaving the bucket) at the hole a is, according to the equation of continuity,

$$\mathbf{V} = \frac{A}{a}\mathbf{v} = -\frac{A}{a}\dot{h}\,\hat{\mathbf{z}} \qquad \left(V = \frac{A}{a}v = -\frac{A}{a}\dot{h}\right). \tag{3}$$

where both velocities are measured in the rest frame of the bucket.

The system of bucket plus water therein has two degrees of freedom, h and z_s, so we seek two equations of motion.

2.1 Energy and Momentum Analysis

In this section we follow the spirit of Bernoulli [2, 3, 4] in using an energy argument to obtain one equation of motion, and then follow Newton to obtain a second equation of motion via a momentum analysis.

2.1.1 Energy Analysis

The bucket has downwards acceleration equal $\ddot{z}_s$, so in the instantaneous (accelerated) rest frame of the bucket the effective gravitational acceleration (downwards) is,

$$g^\star(t) = g - \ddot{z}_s(t). \tag{4}$$

We apply Bernoulli's method (an innovation in 1738) in the rest frame of the bucket, arguing that, in this frame, the rate at which work is done on the water in the bucket by the effective gravitation $g^\star$ is equal to the rate of change of kinetic energy of the water in the bucket plus the rate at which kinetic energy exits the bucket through the hole. That is, the method is based on conservation of energy.

The rate dW/dt of gravitational work on the water in the tank at time t is the product of the rate $\rho g^\star v$ of effective gravitational work per unit volume, and the volume Ah of the water in the bucket,

$$\frac{dW}{dt} = \rho g^\star v A h = \rho g^\star V a h. \tag{5}$$

The total kinetic energy of the water in the bucket (in its rest frame) is the product of the kinetic energy per unit volume $\rho v^2/2$ and by the volume of the water in the tank,

$$\mathrm{KE}_{\mathrm{tank}} = \frac{\rho v^2}{2}Ah = \frac{\rho V^2}{2}\frac{a^2}{A}h, \tag{6}$$

$$\frac{d\,\mathrm{KE}_{\mathrm{tank}}}{dt} = \frac{\rho V^2}{2}\frac{a^2}{A}\frac{dh}{dt} + \rho V\frac{dV}{dt}\frac{a^2}{A}h = -\frac{\rho V^3}{2}\frac{a^3}{A^2} + \rho V\frac{dV}{dt}\frac{a^2}{A}h, \tag{7}$$

using eq. (3) to obtain the last form of eq. (7).

The rate at which kinetic energy exits the bucket (in its rest frame) is given by,

$$\frac{d\,\mathrm{KE}_{\mathrm{exit}}}{dt} = \rho V a\frac{V^2}{2}\,. \tag{8}$$

Conservation of energy now implies that,

$$\frac{dW}{dt} = \rho g^\star V a h = \frac{d\,\mathrm{KE}_{\mathrm{tank}}}{dt} + \frac{d\,\mathrm{KE}_{\mathrm{exit}}}{dt} = -\frac{\rho V^3}{2}\frac{a^3}{A^2} + \rho V \frac{dV}{dt}\frac{a^2}{A} h + \rho V a \frac{V^2}{2}\,. \tag{9}$$

If we divide eq. (9) by $\rho V a$, we obtain,

$$g^\star h = (g - \ddot{z}_s) h = \left(1 - \frac{a^2}{A^2}\right)\frac{V^2}{2} + \frac{a}{A} h \frac{dV}{dt} = -h\ddot{h} + \frac{\dot{h}^2}{2}\left(\frac{A^2}{a^2} - 1\right). \tag{10}$$

Time t can be replaced as the independent variable in this equation by the depth h, by combining the first form of eq. (10) with eq. (3) to yield,[1]

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = -\frac{a}{A} V \frac{dV}{dh} = -\frac{1}{2}\frac{a}{A}\frac{dV^2}{dh}\,, \tag{15}$$

$$2g^\star h = \left(1 - \frac{a^2}{A^2}\right) V^2 - \frac{a^2}{A^2} h \frac{dV^2}{dh}\,. \tag{16}$$

2.1.2 Momentum Analysis

Following Newton, the total force on the system of bucket plus water therein equals the rate of change of momentum of the system, $\mathbf{F}_{\mathrm{tot}} = d\mathbf{p}/dt$. In the inertial lab frame, the force on

[1]The last term in eq. (10) involves the derivative of V^2 with respect to h, which term captures the effect of the fluid acceleration in the tank that is omitted in the steady-flow version of the Bernoulli equation.

There exists a so-called extended Bernoulli equation, which can be applied to examples like the present in which the (incompressible, inviscid) system of interest is in a noninertial frame, and in which the flow is not steady. In this case, the nominal Bernoulli equation is supplemented by a "correction" term obtained by an appropriate integration along the streamline,

$$P_1 + \frac{\rho u_1^2}{2} + \rho g h_1 = P_2 + \frac{\rho u_2^2}{2} + \rho g h_2 + \int_1^2 \text{“correction”}, \qquad \text{(extended Bernoulli)}, \tag{11}$$

where $\mathbf{u}(\mathbf{r}, t) = -\dot{h}\,\hat{\mathbf{z}}$ is the unsteady velocity of the fluid in the system, and the (complicated) "correction" term is displayed in eq. (12) of [9].

In the present example, the "correction" term is,

$$\int_1^2 \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{d^2\mathbf{O}}{dt^2}\right) \cdot d\mathbf{l}, \tag{12}$$

where $\mathbf{O}$ is the origin of the coordinates of the noninertial frame of the system with respect to an inertial lab frame. The origin $\mathbf{O}$ of the coordinate system of the accelerated frame is at $(0, 0, z_s)$, so $d^2\mathbf{O}/dt^2 = \ddot{z}_s\,\hat{\mathbf{z}}$.

Taking point 1 at the center of the upper surface of the water in the bucket ($z_1 = z_s + H - h$), and point 2 at the center of the hole at the bottom of the bucket ($z_2 = z_s + H$), we ignore the tiny difference in atmospheric pressure between these points, and note that $u_1 = -\dot{h} = aV/A$, $u_2 = V$, and,

$$\int_1^2 \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{d^2\mathbf{O}}{dt^2}\right) \cdot d\mathbf{l} = \rho\left(-\ddot{h} + \ddot{z}_z\right) \int_{z_s+H-h}^{z_s+H} dz = -\rho h \frac{d^2 h}{dt^2} + \rho h \ddot{z}_s = \rho \frac{a}{A} h \frac{dV}{dt} + \rho h \ddot{z}_s. \tag{13}$$

Then, eq. (11) becomes, after dividing by ρ,

$$(g - \ddot{z}_s) h = \left(1 - \frac{a^2}{A^2}\right)\frac{V^2}{2} + \frac{a}{A} h \frac{dV}{dt} = \left(1 - \frac{a^2}{A^2}\right)\frac{V^2}{2} - \frac{a^2}{2A^2} h \frac{dV^2}{dh}\,. \tag{14}$$

as in eqs. (10) and (16).

the system consists not only of the external force,

$$\mathbf{F}_{\rm ext} = M_{\rm tot}\mathbf{g} - k\mathbf{z}_s, \qquad F_{{\rm ext},z} = Mg + \rho A h g - k z_s, \tag{17}$$

but also the reaction force of the momentum leaving the system through the hole in the bucket (a kind of "rocket propulsion"),

$$\mathbf{F}_{\rm react} = -\frac{d\mathbf{p}_{\rm leaving}}{dt} = -\frac{dm_{\rm leaving}}{dt}(\dot{\mathbf{z}}_s + \mathbf{V}) = \rho a h\left(\dot{\mathbf{z}}_s - \frac{A}{a}\dot{h}\,\hat{\mathbf{z}}\right), \tag{18}$$

$$F_{{\rm react},z} = \rho a h\left(\dot{z}_s - \frac{A}{a}\dot{h}\right). \tag{19}$$

The momentum of the system, and its rate of change, are,

$$\mathbf{p} = M\dot{\mathbf{z}}_s + \rho A h(\dot{\mathbf{z}}_s - \dot{h}\,\hat{\mathbf{z}}), \qquad \frac{dp_z}{dt} = (M + \rho A h)\ddot{z}_s + \rho A\dot{h}\dot{z}_s - \rho A\dot{h}^2 - \rho A h\ddot{h}. \tag{20}$$

The Newtonian equation of motion of the system is, hence,

$$\rho A\dot{h}^2\left(\frac{A}{a} - 1\right) - \rho A h\ddot{h} = (M + \rho A h)(g - \ddot{z}_s) - k z_s. \tag{21}$$

Recalling the last form of eq. (10) for $(g - \ddot{z}_s)h$, we can rewrite eq. (21) as,[2]

$$\rho A\dot{h}^2\left(\frac{A}{a} - 1\right) - \rho A h\ddot{h} = M(g - \ddot{z}_s) + \rho A\left[-h\ddot{h} + \frac{\dot{h}^2}{2}\left(\frac{A^2}{a^2} - 1\right)\right] - k z_s. \tag{23}$$

$$\rho A\dot{h}^2\left[\frac{1}{2} + \frac{A}{a}\left(\frac{A}{2a} - 1\right)\right] = k z_s - M(g - \ddot{z}_s). \tag{24}$$

2.1.3 $a = A$

For the limiting case that $a = A$, the water just falls free of the bucket,[3] and eq. (24) becomes the equation of motion for the bucket in the absence of any water,

$$M\ddot{z}_s = Mg - k z_s, \quad \ddot{z}_s = -\omega^2 z_s + g, \quad z_s = \frac{Mg}{k} + \left(z_{s0} - \frac{Mg}{k}\right)\cos\omega_0 t + \frac{\dot{z}_{s0}}{\omega}\sin\omega_0 t, \tag{25}$$

where $\omega_0^2 = k/M$.

For example, if the bucket plus water were initially at rest, $z_{s0} = (M + \rho a h_0)/k$ and $\dot{z}_{s0} = 0$, and the bottom of the bucket were somehow removed at time $t = 0$, the subsequent oscillation of the bucket would be described by,

$$z_s = \frac{Mg}{k} + \frac{\rho A h_0}{k}\cos\omega_0 t. \tag{26}$$

---

[2] March 3, 2023. If instead of a spring, the system were subject to a vertical force F (positive downwards), this would replace the force $-k z_s$ in eqs. (17), (21) and (23)-(24). Then, eq. (24) would be,

$$\rho A\dot{h}^2\left[\frac{1}{2} + \frac{A}{a}\left(\frac{A}{2a} - 1\right)\right] = -F - M(g - \ddot{z}_s). \tag{22}$$

This equation does not depend on the vertical velocity $\dot{z}_s$, and hence the system is invariant under a (Galilean) transformation in coordinate z. Such invariance does not generally hold for systems in which some form of Bernoulli's equation is relevant [5, 6, 7].

[3] The top surface of the falling water is at $z_{\rm top} = z_{s0} + H - h_0 + (\dot{z}_{s0} - \dot{h}_0)t + gt^2/2$.

2.1.4 $a \ll A$

In this case, eq. (16) becomes,

$$2g^\star h = 2(g - \ddot{z}_s)h \approx V^2 = \frac{A^2}{a^2}\dot{h}^2, \tag{27}$$

and with this, eq. (24) becomes,

$$\rho A h(g - \ddot{z}_s) \approx k z_s - M(g - \ddot{z}_s), \qquad \ddot{z}_s \approx -\omega^2 z_s + g, \tag{28}$$

$$z_s \approx \frac{M_{\rm tot} g}{k} + \left(z_{s0} - \frac{M_{\rm tot} g}{k}\right)\cos\omega t + \frac{\dot{z}_{s0}}{\omega}\sin\omega t, \tag{29}$$

where $\omega^2 = k/M_{\rm tot} = k/(M + \rho A h)$ increases slowly with time as the water drains out of the bucket. Then, eq. (27) can be rewritten as,

$$V^2 = \frac{A^2}{a^2}\dot{h}^2 \approx 2(g - \ddot{z}_s)h \approx 2\omega^2 h z_s \approx 2gh, \tag{30}$$

and V^2 oscillates about the value $2gh(t)$, which would be its value if the bucket were at rest. The water has completely drained from the bucket after a time that is approximately the same as if the bucket remained at rest, namely,

$$\dot{h} \approx -\frac{a}{A}\sqrt{2gh}, \qquad \sqrt{h} \approx \sqrt{h_0} - \frac{a}{A}\sqrt{\frac{g}{2}}t, \qquad t_{\rm drain} \approx \frac{A}{a}\sqrt{\frac{2h_0}{g}}. \tag{31}$$

For example, if the bucket plus water were initially at rest, $z_{s0} = (M + \rho a h_0)/k$ and $\dot{z}_{s0} = 0$, and the small hole were opened at time $t = 0$, the subsequent oscillation of the bucket would be described by,[4]

$$z_s \approx \frac{Mg}{k} + \frac{\rho A h_0}{k}\cos\omega t. \tag{32}$$

2.1.5 Motion When $M \ll \rho A h$

The motion in the general case of $0 < a/A < 1$ consists of an oscillation in z_s as the water level h decrease with time. We don't pursue analytic description of the general motion further, but we note that if the mass $\rho A h$ of the water in the bucket is large compared to the mass M of the bucket, eq. (24) simplifies to,

$$\rho A \dot{h}^2 \left[\frac{1}{2} + \frac{A}{a}\left(\frac{A}{2a} - 1\right)\right] = k z_s, \tag{33}$$

so that $z_s \propto \dot{h}^2 \propto V^2$, while all of these quantities oscillate in time. However, when the mass of the water in the bucket is small compared to M, the correlation of z_s with $\dot{h}^2$ no longer holds.

[4]The case $a \ll A$ has been discussed in [8].

2.2 A Lagrangian Approach

A Lagrangian approach to variable-mass problems has been given in [10, 11].

For the present example, it seems appropriate to consider the system to be only the bucket plus water therein, which can be characterized by two coordinates, z_s and h. The velocity V of the efflux of water from the bucket is related, in the rest frame of the bucket, by the continuity equation for incompressible fluids, as in eq. (4) above,

$$V = \frac{av}{A} = -\frac{a\dot{h}}{A}\,. \tag{34}$$

The kinetic energy of the system is,

$$T = \frac{M\dot{z}_s^2}{2} + \frac{\rho A h(\dot{z}_s - \dot{h})^2}{2}\,. \tag{35}$$

While one can give an expression for the gravitational potential energy of this system, the force on the system is not simply related to this potential energy, so the latter is not used in the method of [10]. Rather, one uses generalized forces, Q_{z_s} and Q_h, as introduced by Lagrange.

We recall that for a system with a set of coordinates q_k (which could be functions of time t) and kinetic energy $T(q_k, \dot{q}_k, t)$, Lagrange's equations can be written as,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k = \frac{\partial}{\partial q_k}\sum_i \mathbf{F}_i^{\rm ext}\cdot \mathbf{r}_i \equiv \frac{\partial U}{\partial q_k}\,, \tag{36}$$

where $\mathbf{r}_i$ is the (x, y, z) coordinate of the $i^{\rm th}$ particle in the system, $\mathbf{F}_i^{\rm ext}$ is the external force on particle i, and $U = \sum_i \mathbf{F}_i^{\rm ext}\cdot \mathbf{r}_i$ was called the force function by Hamilton, p. 249 of [12], following Lagrange [13], who built on a principle of d'Alembert [14].

In a variable-mass problem such as the present example, the flow of water out of the hole in the bucket is associated with a reaction force on the water still in the bucket. In the Newtonian approach, this reaction force must be included in the equation(s) of motion (sec. 2.1.2 above), but in Lagrangian approach the reaction force is not considered to be an external force, and so is not to be included in the generalized forces.

In the present example, a molecule i of the bucket has position $\mathbf{r}_{\mathrm{bucket},i} = (x_i, y_i, z_s + \Delta z_i)$, and a molecule j of the water in the bucket has position $\mathbf{r}_{\mathrm{water},j} = (x_j, y_j, z_s + H - h_j)$. The external force on a molecule of the bucket is $\mathbf{F}^{\rm ext}_{\mathrm{bucket},i} = -k\mathbf{z}_s + m_{\mathrm{mol},i}\, g\,\hat{\mathbf{z}}$, due to the spring and to gravity, but we (delicately) consider that the spring does not exert an external force on molecules of water, such that $\mathbf{F}^{\rm ext}_{\mathrm{water},j} = m_{\mathrm{mol},j}\, g\,\hat{\mathbf{z}}$, due only to gravity.[5] Then, the generalized force Q_h is given by,

$$\begin{aligned} Q_h &= \frac{\partial}{\partial h}\left(-\sum_i k\mathbf{z}_s\cdot\mathbf{r}_i + \sum_i m_{\mathrm{mol}_i}\, g\,\hat{\mathbf{z}}\cdot\mathbf{r}_i + \sum_j m_{\mathrm{mol}_j}\, g\,\hat{\mathbf{z}}\cdot\mathbf{r}_j\right) \\ &= \sum_j m_{\mathrm{mol}_j}\, g\,\hat{\mathbf{z}}\cdot(-\hat{\mathbf{z}}) = -mg = -\rho A g h. \end{aligned} \tag{37}$$

[5]In the present approximation, the bucket is a rigid body, such that while the spring is connected to the bucket at a single point, we consider that the (external) spring force acts on the entire bucket. The bottom of the bucket exerts a normal force on the water above, which we consider to be an internal force, not to be included in the generalized force.

Similarly, the generalized force Q_{z_s} is given by,

$$
\begin{aligned}
Q_{z_s} &= -\sum_i k\mathbf{z}_s \cdot \frac{\partial \mathbf{r}_i}{\partial z_s} + \sum_j m_{\mathrm{mol}_j}\, g\, \hat{\mathbf{z}} \cdot \frac{\partial \mathbf{r}_j}{\partial z_s} + \sum_j m_{\mathrm{mol}_j}\, g\, \hat{\mathbf{z}} \cdot \frac{\partial \mathbf{r}_j}{\partial z_s} \\
&= -\sum_i k\mathbf{z}_s \cdot \hat{\mathbf{z}} + \sum_j m_{\mathrm{mol}_j}\, g\, \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} + \sum_j m_{\mathrm{mol}_j}\, g\, \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = -kz_s + (M + \rho Ah)g. \qquad (38)
\end{aligned}
$$

In the method of [10, 11], the left side of eq. (36) is modified for a variable-mass system, whose (control) volume has velocity $\mathbf{w}$, according to eq. (5.6) of [10] and eq. (1) of [11],[6]

$$
\frac{d}{dt}\frac{\partial T_w}{\partial \dot{q}_k} - \frac{\partial T_w}{\partial q_k} + \int \frac{\partial \tilde{T}}{\partial \dot{q}_k}(\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} - \int \tilde{T} \frac{\partial (\mathbf{v} - \mathbf{w})}{\partial \dot{q}_k} \cdot d\mathbf{Area} = Q_k, \qquad (39)
$$

where T_w is the kinetic energy within the control volume, $\tilde{T}$ is the kinetic energy per unit volume, and $\mathbf{v}$ is the velocity of the material at a point in the system.

In the present example, the control volume is the bucket and water therein, so $\mathbf{w} = \dot{\mathbf{z}}_s$, T_w is given in eq. (35), and inside the control volume $\mathbf{v} = \dot{\mathbf{z}}_s - \dot{h}\,\hat{\mathbf{z}}$ and $\tilde{T} = \rho_{\mathrm{bucket}}\, \dot{z}_s^2/2 + \rho(\dot{z}_s - \dot{h})^2/2$. However, for eq. (39) we must consider the surface of the control volume, where $\mathbf{v} = \mathbf{w}$ except at the hole, at which $\mathbf{v} - \mathbf{w} = \mathbf{V} = V\,\hat{\mathbf{z}} = -A\dot{h}\,\hat{\mathbf{z}}/a$ and,

$$
\tilde{T} = \frac{\rho(\dot{z}_s + V)^2}{2} = \frac{\rho}{2}\left(\dot{z}_s^2 - 2\frac{A}{a}\dot{z}_s\dot{h} + \frac{A^2}{a^2}\dot{h}^2\right) \qquad \text{(hole)}. \qquad (40)
$$

2.2.1 Equation of Motion for Coordinate h

Recalling eq. (35),

$$
\frac{d}{dt}\frac{\partial T_w}{\partial \dot{h}} = \rho Ah(\ddot{h} - \ddot{z}_s) + \rho A\dot{h}^2, \qquad \frac{\partial T_w}{\partial h} = \frac{\rho A(\dot{z}_s - \dot{h})^2}{2}, \qquad (41)
$$

and at the hole, where the area vector is direction outwards, with $d\mathbf{Area} = a\,\hat{\mathbf{z}}$,

$$
\frac{\partial \tilde{T}}{\partial \dot{h}} = \rho\dot{h}\frac{A^2}{a^2} - \rho\frac{A}{a}\dot{z}_s, \qquad \mathbf{v} - \mathbf{w} = \mathbf{V} = -\frac{A}{a}\dot{h}\,\hat{\mathbf{z}}, \qquad \frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{h}} = -\frac{A}{a}\,\hat{\mathbf{z}}. \qquad (42)
$$

Hence, the equation of motion (39) for the coordinate h is,

$$
\begin{aligned}
&\rho Ah(\ddot{h} - \ddot{z}_s) + \rho A\dot{h}^2 - \frac{\rho A}{2}(\dot{z}_s^2 - 2\dot{z}_s\dot{h} + \dot{h}^2) - \rho A\dot{h}\left(\dot{h}\frac{A^2}{a^2} - \frac{A}{a}\dot{z}_s\right) \\
&+\frac{\rho A}{2}\left(\dot{z}_s^2 - 2\frac{A}{a}\dot{z}_s\dot{h} + \frac{A^2}{a^2}\dot{h}^2\right) = -\rho Agh, \qquad (43) \\
&h(\ddot{h} - \ddot{z}_s) - \left(\frac{A^2}{a^2} - 1\right)\frac{\dot{h}^2}{2} = -gh. \qquad (44)
\end{aligned}
$$

as previously found in eq. (16).

[6] An earlier discussion of Lagrange's equations for systems of variable mass was given in [15] (1947), where the context was rocket motion. It was noted that although the system of rocket plus fuel has variable mass, the center of mass of this system remains constant to a reasonable approximation, relative to the system, which permits a simpler form of the equations of motion than eq. (39).

2.2.2 Equation of Motion for Coordinate z_s

From eq. (35),

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{z}_s} = M\ddot{z}_s - \rho A h(\ddot{h} - \ddot{z}_s) - \rho A\dot{h}(\dot{h} - \dot{z}_s), \qquad \frac{\partial T_w}{\partial z_s} = 0, \tag{45}$$

and at the hole, where the area vector is direction outwards, with $d\mathbf{Area} = a\,\hat{\mathbf{z}}$,

$$\frac{\partial \tilde{T}}{\partial \dot{z}_s} = \rho\dot{z}_s - \rho\frac{A}{a}\dot{h}, \qquad \mathbf{v} - \mathbf{w} = \mathbf{V} = -\frac{A}{a}\dot{h}\,\hat{\mathbf{z}}, \qquad \frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{z}_s} = 0. \tag{46}$$

Hence, the equation of motion (39) for the coordinate z_s is,

$$M\ddot{z}_s - \rho A h(\ddot{h} - \ddot{z}_s) - \rho A\dot{h}(\dot{h} - \dot{z}_s) - \rho A\dot{h}\left(\dot{z}_s - \frac{A}{a}\dot{h}\right) = -kz_s + (M + \rho A h)g, \tag{47}$$

$$\left(\frac{A}{a} - 1\right)\frac{\rho A\dot{h}^2}{2} - \rho A h\ddot{h} = -kz_s + (M + \rho A h)(g - \ddot{z}_s), \tag{48}$$

as previously found in eq. (21).

References

[1] E. Torricelli, *Opera Geometrica* (Florence, 1644),
http://kirkmcd.princeton.edu/examples/mechanics/torricelli_opera_44.pdf
http://kirkmcd.princeton.edu/examples/mechanics/torricelli_opera_p191.pdf

[2] J. Otto and K.T. McDonald, *Torricelli's Law for Large Holes* (May 15, 2018),
http://kirkmcd.princeton.edu/examples/leaky_tank.pdf

[3] D. Bernoulli, *Hydrodynamica* (Strassburg, Austria, 1738),
http://kirkmcd.princeton.edu/examples/fluids/bernoulli_hydrodynamica_38.pdf
Hydrodynamics by Daniel Bernoulli and Hydraulics by Johann Bernoulli, trans. by T. Carmody and H. Kobus (Dover, 1968),
http://kirkmcd.princeton.edu/examples/fluids/bernoulli_ch3_13.pdf

[4] A. Malcherek, *History of the Torricelli Principle and a New Outflow Theory*, J. Hydraul. Eng. **142**, 02516004 (2016),
http://kirkmcd.princeton.edu/examples/fluids/malcherek_jhe_142_02516004_16.pdf

[5] K.T. McDonald, *Is Bernoulli's Equation Relativistically Invariant?* (Sept. 13, 2009),
http://kirkmcd.princeton.edu/examples/bernoulli.pdf

[6] C.E. Mungan, *The Bernoulli equation in a moving reference frame*, Eur. J. Phys. **32**, 517 (2011), http://kirkmcd.princeton.edu/examples/fluids/mungan_ejp_32_517_11.pdf

[7] A. Maranzoni, *Galilean-Invariant Expression for Bernoulli's Equation*, J. Hydr. Eng. **146**, 04019061 (2020),
http://kirkmcd.princeton.edu/examples/fluids/maranzoni_jhe_146_04019061_20.pdf

[8] H. Rodrigues *et al.*, *The motion of a leaking oscillator: a study for the physics class*, Phys. Ed. **49**, 557 (2014), http://kirkmcd.princeton.edu/examples/mechanics/rodrigues_pe_49_557_14.pdf

[9] S.B. Segletes and W.P. Walters, *A note on the application of the extended Bernoulli equation*, Int. J. Imp. Eng. **27**, 561 (2002), http://kirkmcd.princeton.edu/examples/fluids/segletes_ARL-TR-1895_99.pdf

[10] H. Irschik and H.J. Holl, *The equations of Lagrange written for a non-material volume*, Acta Mech. **153**, 231 (2002), http://kirkmcd.princeton.edu/examples/mechanics/irschik_am_153_231_02.pdf

[11] H. Irschik and H.J. Holl, *Lagrange's equations for open systems, derived via the method of fictitious particles, and written in the Lagrange description of continuum mechanics*, Acta Mech. **226**, 63 (2015), http://kirkmcd.princeton.edu/examples/mechanics/irschik_am_226_63_15.pdf

[12] W.R. Hamilton, *On a General Method in Dynamics* Phil. Trans. Roy. Soc. London **124**, 247 (1834), http://kirkmcd.princeton.edu/examples/mechanics/hamilton_ptrsl_124_247_34.pdf

[13] J.-L. Lagrange, *Méchanique Analitique*, (DeSaint, Paris, 1788), http://kirkmcd.princeton.edu/examples/mechanics/lagrange_v1_13.pdf

[14] J.-B. D'Alembert, *Trait de Dynamique* (David, Paris, 1743), http://kirkmcd.princeton.edu/examples/mechanics/dalembert_dynamique.pdf

[15] D.F. Lawden, *Lagrange's Equations for a System of Variable Mass*, J. Brit. Interplanetary Soc. **8**, 174 (1947), http://kirkmcd.princeton.edu/examples/mechanics/lawden_jbis_8_174_47.pdf