Vertical Oscillations of a Hanging Cable

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 7, 2001)

1 Problem

Deduce an expression for the possible frequencies of oscillation of a cable of mass m, length l and spring constant k when its upper end is fixed, and mass M is suspended from its lower end. What is the lowest angular frequency of oscillation for the special cases that a) $M = 0$, b) $m = 0$, and c) $m/M \ll 1$?

2 Solution

The force of gravity and the constraint force at the upper end of the cable determine the form of the static solution, but do not enter into a description of small oscillations (for which the quadratic spring potential is what matters).

Before discussing the general case, we briefly examine special cases b) and c).

Case b) when the cable has zero mass is, of course, that of a mass M hanging from a spring of constant k. The angular frequency of oscillation is,

$$\omega = \sqrt{\frac{k}{M}} \qquad \text{(Case b)}. \tag{1}$$

The lowest frequency of oscillation in case c) can be deduced by an application of the virial theorem,[1] which says that in bounded motion due to a quadratic potential, the time-average kinetic and potential energies are equal. For the lowest mode when $m \ll M$, a point whose unstretched position is x (taken to be positive downwards, with $x = 0$ at the upper end of the cable) has displacement,

$$s(x,t) = (A + B\cos\omega t)\frac{x}{l}\,, \tag{2}$$

where A is the static displacement and B is the amplitude of oscillation of the lower end of the cable. The time-average kinetic energy of the oscillating cable plus mass M is,

$$\langle KE\rangle = \frac{1}{2}B^2\omega^2\left\langle\sin^2\omega t\right\rangle\left(M + m\int_0^l \frac{x^2}{l^2}\frac{dx}{l}\right) = \frac{1}{4}B^2\omega^2\left(M + \frac{m}{3}\right). \tag{3}$$

The time-average of the spring potential energy of the system is,

$$\langle PE\rangle = \frac{1}{2}kB^2\left\langle\cos^2\omega t\right\rangle = \frac{1}{4}kB^2. \tag{4}$$

[1] See, for example, sec. 3.4 of H. Goldstein, C.P. Poole and H. Safko, *Classical Mechanics*, 3$^{\rm rd}$ ed. (Addison-Wesley, 2002), http://kirkmcd.princeton.edu/examples/mechanics/goldstein_3ed.pdf

Equating the average kinetic and potential energies we find,

$$\omega = \sqrt{\frac{k}{M + m/3}} \qquad \text{(Case c)}. \tag{5}$$

We now take up the analysis of the general case.

We define $f(x)$ as the internal force across a horizontal plane through point x. [f, like x, is positive downwards.] The equation of motion for the displacement $s(x,t)$ of an element dx of the cable, centered on x, is then,

$$m\frac{dx}{l}\ddot{s} = f(x+dx) - f(x) + m\frac{dx}{l}g = \left(f'(x) + \frac{mg}{l}\right)dx, \tag{6}$$

so that,

$$\ddot{s} = \frac{l}{m}f'(x) + g. \tag{7}$$

Due to the internal force f the element dx has stretched by amount,

$$s(x+dx) - s(x) = s'(x)dx. \tag{8}$$

We recall that stretching of an elastic medium can be related to its elastic modulus E via,

$$\frac{f}{A} = E\frac{\Delta l}{l}, \tag{9}$$

where A is the area of a cross section perpendicular to the direction of the stretch. Considering the entire cable, we see that the modulus E and the spring constant k are related by,

$$k = \frac{EA}{l}. \tag{10}$$

For the element dx eq. (9) becomes,

$$f = EA\frac{s'dx}{dx} = kls'. \tag{11}$$

Inserting this in the equation of motion (7) we find the wave equation,

$$\ddot{s} = \frac{kl^2}{m}s'' + g. \tag{12}$$

The boundary condition at the upper end of the cable is simply,

$$s(0,t) = 0. \tag{13}$$

At the lower end of the cable, the equation of motion of the mass M is,

$$M\ddot{s}(l,t) = -f(l) + Mg = -kls'(l,t) + Mg, \tag{14}$$

noting that $f(x)$ is the downward force in the cable at x while we need the upward force here.

We first display the static solution s_0 for which eq. (12) becomes,

$$s_0'' = -\frac{mg}{kl^2}. \tag{15}$$

Integrating once, and using condition (14) with $\ddot{s}(l,t) = 0$, we find,

$$s_0' = \frac{mg}{kl}\left(1 - \frac{x}{l}\right) + \frac{Mg}{kl}. \tag{16}$$

Integrating again, and using condition (13), we find,

$$s_0(x) = \frac{mg}{kl}\left(x - \frac{x^2}{2l}\right) + \frac{Mgx}{kl}. \tag{17}$$

We now consider an oscillatory displacement $s_1(x,t)$ in addition to the static displacement s_0. That is, $s(x,t) = s_0(x) + s_1(x,t)$. The equation of motion for s_1 follows from eq. (12) as,

$$\ddot{s}_1 = \frac{kl^2}{m} s_1''. \tag{18}$$

The boundary condition at $x = 0$ remains eq. (13), while at the lower end of the cable eq. (14) tells us that,

$$M\ddot{s}_1 = -kls_1'. \tag{19}$$

As anticipated, gravity does not enter into the description of oscillatory motion.

We seek standing wave solutions of frequency ω,

$$s_1 = g(x)\cos\omega t. \tag{20}$$

Inserting this in eq. (18) we find,

$$g'' = -\frac{m\omega^2}{kl^2} g, \tag{21}$$

which is solved by,

$$g = A \sin\sqrt{\frac{m}{k}}\frac{\omega x}{l}, \tag{22}$$

in view of boundary condition (13). Finally, the condition (19) tells us thatk

$$M\omega^2 \sin\sqrt{\frac{m}{k}}\omega = kl\sqrt{\frac{m}{k}}\frac{\omega}{l}\cos\sqrt{\frac{m}{k}}\omega, \tag{23}$$

ork

$$\sqrt{\frac{m}{k}}\omega \tan\sqrt{\frac{m}{k}}\omega = \frac{m}{M}. \tag{24}$$

This is the general solution to the problem.

If $M = 0$, then $\tan\omega\sqrt{m/k} = \infty$, so that

$$\omega = \frac{(2n+1)\pi}{2}\sqrt{\frac{k}{m}} \qquad \text{(Case a)}. \tag{25}$$

If $m = 0$, the cable is massless and eq. (21) reverts to $g'' = 0$, which is solved by $g = Ax/l$. The boundary condition (19) then yields,

$$\omega = \sqrt{\frac{k}{M}} \qquad \text{(Case b)}. \tag{26}$$

If $m/M \ll 1$, then according to eq. (24) the lowest frequency occurs for a small value of $\omega\sqrt{m/k}$. We recall that for a small argument the tangent is approximately,

$$\tan\epsilon = \frac{\sin\epsilon}{\cos\epsilon} \approx \frac{\epsilon - \epsilon^3/6}{1 - \epsilon^2/2} \approx \epsilon + \frac{\epsilon}{3}\,. \tag{27}$$

Then, eq. (24) becomes,

$$\frac{m}{k}\omega^2\left(1 + \frac{m}{k}\frac{\omega^2}{3}\right) \approx \frac{m}{M}\,. \tag{28}$$

Hence,

$$\omega^2 \approx \frac{k}{M(1 + m\omega^2/3k)}\,. \tag{29}$$

Replacing ω^2 on the righthand side of eq. (29) by its first approximation k/M, we find the next approximation,

$$\omega^2 \approx \frac{k}{M + m/3} \qquad \text{(Case c)}, \tag{30}$$

in agreement with the result (5) obtained using the virial theorem.