Beer Can in Orbit about a Space Station

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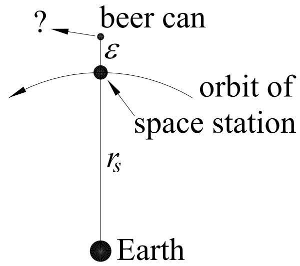
1 Problem

A space station is in a circular orbit of radius r_s about the Earth. An astronaut is on a spacewalk at distance $r_s + \epsilon$, with $\epsilon \ll r_s$, from the center of the Earth along the radius through the space station. With practice, the astronaut can throw a "beer can" so that it moves in the plane of the orbit of the space station, and appears to orbit the space station.



In what direction, and with what speed should the astronaut throw the beer can? What is the shape, size and period of the orbit of the beer can?

2 Solution

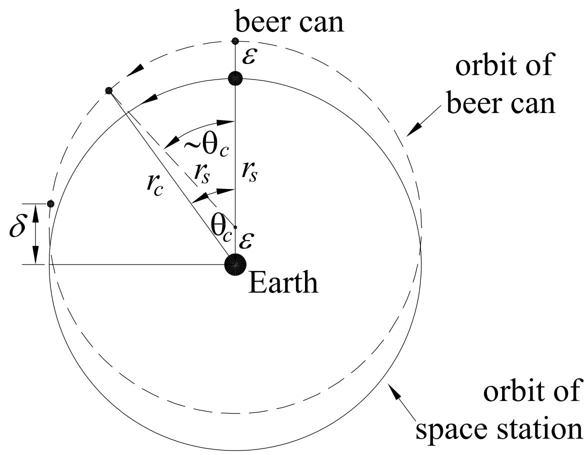
2.1 Analysis in the Rest Frame of the Earth

We neglect the effect of gravity between the space station and beer can (and also that of the Sun and the various other objects in the Solar system).

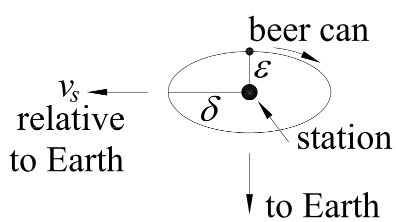
Then, the orbit of the beer can is an ellipse of eccentricity $\epsilon_c = \epsilon/r_s$ about the Earth, which orbit is very close to the orbit of the space station. For small ϵ the orbit of the beer can is essentially a circle of radius r_s, *i.e.*, an orbit with semimajor axis $a \approx r_s$, and period $T_s = 2\pi/\omega_s = 2\pi\sqrt{r_s^3/Gm_E}$ of the space station,[1]

$$\frac{1}{r_c} = \frac{1 - \epsilon_c \cos\theta_c}{a(1 - \epsilon_c^2)}, \qquad r_c \approx r_s(1 + \epsilon_c \cos\theta_c) = r_s + \epsilon \cos\omega_s t, \tag{1}$$

where to a first approximation, $\theta_c = \omega_s t$.



When the space station has advanced by 90° in its orbit about the Earth, with respect to the configuration in the first figure on p. 1, we see that the beer can is behind the space station by distance δ. This implies that relative to the space station, the motion of the beer can is retrograde compared to the motion of the space station about the Earth.



While $\theta_c \approx \omega_s t$, which is a sufficient approximation for eq. (1), to determine δ and the shape of the orbit of the beer can relative to the space station, we need a better approximation. For this, we note that the angular momentum L_c of the beer can about the Earth is constant (ignoring gravitational effects of the Sun, Moon, space station, *etc.*),

$$\dot{\theta}_c = \frac{L_c}{m_c r_c^2} \approx \frac{L_c}{m_c r_s^2 (1 + \frac{\epsilon}{r_s \omega_s} \sin\omega_s t)^2} \approx \frac{L_c}{m_c r_s^2}\left(1 - 2\frac{\epsilon}{r_s}\cos\omega_s t\right). \tag{2}$$

[1] This approximation is reviewed on p. 98 of Ph205 Lecture 9, http://kirkmcd.princeton.edu/examples/Ph205/ph205l9.pdf

We can integrate eq. (2),

$$\theta_c \approx \frac{L_c}{m_c r_s^2}\left(t - 2\frac{\epsilon}{r_s\omega_s}\sin\omega_s t\right) = \omega_s t - 2\frac{\epsilon}{r_s}\sin\omega_s t, \tag{3}$$

noting that when $\theta_c = 90^\circ$, $r_c \approx r_s$ and $\dot\theta_c \approx \omega_s$, so $L_c = m_c r_c^2\dot\theta_c \approx m_c r_s^2\omega_s$.

At time $t = 0$, $\dot\theta_c(0) \approx \omega_s - 2\epsilon\omega_s/r_s$, and the velocity of the beer can is parallel to that of the space station, with,

$$v_c(0) = (r_s + \epsilon)\dot\theta_c(0) \approx r_s\omega_s - \epsilon_s\omega_s. \tag{4}$$

The initial velocity of the beer can relative to that of the space station ($v_s = r_s\omega_s$) is,

$$\Delta v_c(0) = v_c(0) - v_s \approx -\epsilon_s\omega_s = -\frac{\epsilon v_s}{r_s} \qquad \text{(relative to space station).} \tag{5}$$

If we suppose that the astronaut moves so as to stay at distance ϵ from the space station along the radius through the center of the space station, then $v_a = (r_s + \epsilon)\omega_s$, parallel to the orbit of the space station, and the initial velocity of the beer can relative to the astronaut is,

$$\Delta v_c(0) = v_c(0) - v_a \approx -2\epsilon_s\omega_s = -2\frac{\epsilon v_s}{r_s} \qquad \text{(relative to astronaut).} \tag{6}$$

For example, with $\epsilon = 10$ m and a space station with period of 2 hours, $\omega_s = 2\pi/7200 \approx 1/600$ rad/s, and $\epsilon\omega_s \approx 1/60$ m/s ≈ 1.5 cm/s.

Finally, to determine δ, we note that in eq. (3), when $\theta_s = \omega_s t = 90^\circ$, $\theta_c \approx \theta_s - 2\epsilon/r_s$, and,

$$\delta = r_s[\theta_s(90^\circ) - \theta_c(90^\circ)] = 2\epsilon. \tag{7}$$

The orbit of the beer can relative to the space station is an ellipse with semimajor axis δ twice its semiminor axis ϵ.

2.2 Analysis in the Rotating Frame of the Space Station

We now consider a rotating frame, centered on the Earth, with angular velocity $\boldsymbol{\omega}_s = \omega_s\,\hat{\mathbf{z}}$ with respect to the nonrotating frame. The space station is at $\mathbf{r}_s = (x, y, z) = (0, r_s, 0)$ in the rotating frame, and the can moves in the x-y plane at position $\mathbf{r}_c = r_s\,\hat{\mathbf{y}} + \mathbf{r}$ with $\mathbf{r} \equiv (x, y, 0)$ in the x-y plane.[2] Then, $\mathbf{F} = m\mathbf{a}$ for the can in the rotating frame includes the centrifugal and Coriolis forces,

$$m\mathbf{a}_c = \mathbf{F} - m\boldsymbol{\omega}_s \times (\boldsymbol{\omega}_s \times \mathbf{r}_c) - 2m\boldsymbol{\omega}_s \times \mathbf{v} = \mathbf{F} + m\omega_s^2\mathbf{r}_c - 2m\omega_s\,\hat{\mathbf{z}} \times (\dot x\,\hat{\mathbf{x}} + \dot y\,\hat{\mathbf{y}}), \tag{8}$$

where $\mathbf{v} = (\dot x, \dot y, 0)$ is the velocity of the can in the rotating frame, the force of gravity from the Earth on the can is,

$$\mathbf{F} = -\frac{Gm_E m\,\hat{\mathbf{r}}_c}{r_c^2} \approx -\frac{Gm_E m\,\hat{\mathbf{y}}}{(r_s + y)^2} \approx -\frac{Gm_E m\,\hat{\mathbf{y}}}{r_s^2}\left(1 - \frac{2y}{r_s}\right), \tag{9}$$

[2] There is some possibility of confusion here, since y is not the y-coordinate of the can relative to the Earth, but relative to the space station.

and $Gm_E m/r_s^2 = m\omega_s^2 r_s$ according to Kepler's third law. Then, eq. (8) can be written as,

$$m\mathbf{a}_c = m(\ddot{x}\,\hat{\mathbf{x}} + \ddot{y}\,\hat{\mathbf{y}}) \approx -m\omega_s^2 r_s \left(1 - \frac{2y}{r_s}\right)\hat{\mathbf{y}} + m\omega_s^2(r_s + y)\,\hat{\mathbf{y}} + 2m\omega_s(\dot{y}\,\hat{\mathbf{x}} - \dot{x}\,\hat{\mathbf{y}})$$
$$= 3m\omega_s^2 y\,\hat{\mathbf{y}} + 2m\omega_s(\dot{y}\,\hat{\mathbf{x}} - \dot{x}\,\hat{\mathbf{y}}). \qquad (10)$$

We try an ellipse-like solution for the initial conditions $x(0) = 0$, $y(0) = \epsilon$,

$$x = \delta \sin \omega t, \qquad y = \epsilon \cos \omega t. \qquad (11)$$

From eq. (10) we have,

$$\ddot{x} = 2\omega_s \dot{y}, \qquad \Rightarrow \qquad -\delta\omega^2 = -2\omega_s\epsilon\omega \qquad \Rightarrow \qquad \delta = \frac{2\omega_s\epsilon}{\omega}, \qquad (12)$$

$$\ddot{y} = 3\omega_s^2 y + 2\omega_s \dot{x} \qquad \Rightarrow \qquad -\epsilon\omega^2 = 3\omega_s\epsilon - 2\omega_s\omega\delta = 3\omega_s\epsilon - 4\omega_s\epsilon = -\omega_s\epsilon \qquad (13)$$

Hence,

$$\omega = \omega_s, \qquad \text{and} \qquad \delta = 2\epsilon. \qquad (14)$$

Thus, we have found the orbit of the can relative to the space station to be the same as that found in sec. 2.1 above. As before,

$$\dot{x}(0) = 2\epsilon\omega_s = 2\epsilon\sqrt{\frac{Gm_E}{r_s^2}}, \qquad \dot{y}(0) = 0, \qquad (15)$$

so the initial motion of the can relative to the space station is with velocity antiparallel to the velocity of the space station relative to the Earth.