Reactance of Small Antennas

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1 Problem

Estimate the capacitance C and self inductance L of a short, center-fed, linear dipole antenna whose arms each have length h and radius a. Also estimate the self inductance of a small loop antenna of major radius b and minor radius a.

For completeness, consider also the real part, its so-called radiation resistance, $R_{\rm rad}$, *of the antenna impedance in the approximation of perfect conductors.*

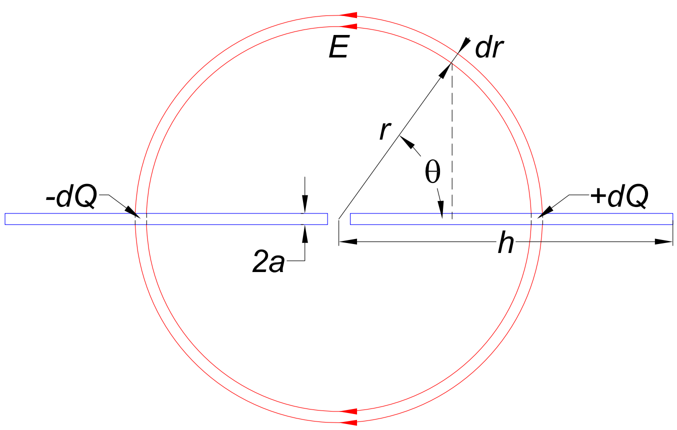
2 Solution

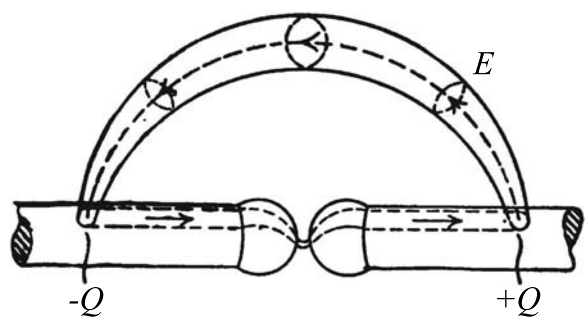
2.1 Short, Center-Fed, Linear Dipole Antenna

This solution follows sec. 10.3 of [1], which can be traced back to [2].

2.1.1 Capacitance

The key assumption is that the electric field lines from one arm of the dipole antenna to the other follow semicircular paths (the principal mode), as shown in the figure below.[1]





If so, all the field lines emanating from charge dQ in interval dr at distance r from the center of the antenna cross a surface of area $2\pi r\, dr\, \sin\theta$ that lies on a cone of half angle θ, so the electric field strength at (r, θ) is,

$$E = \frac{dQ/dr}{2\pi\epsilon_0 r \sin\theta}\,. \qquad (1)$$

[1]On the right is Fig. 86 by Poincaré [2].

The voltage difference between the two arms of the antenna is,[2]

$$\Delta V = 2\int_{\theta_{\min}}^{\pi/2} Er\,d\theta = \frac{dQ/dr}{\pi\epsilon_0}\int_{a/r}^{\pi/2}\frac{d\theta}{\sin\theta} = \frac{dQ/dr}{\pi\epsilon_0}\ln[\tan(\theta/2)]_{a/r}^{\pi/2} = \frac{dQ/dr}{\pi\epsilon_0}\ln(2r/a). \quad (2)$$

This voltage difference should be independent of position along the antenna.[3] The charge distribution dQ/dr is indeed constant to a good approximation for short dipole antennas, but the factor $\ln(2r/a) = -\ln(\theta_{\min}/2)$ is constant only for a biconical dipole antenna (as much favored theoretically by Schelkunoff [1]). A reasonable approximation for a linear dipole antenna is to use $r = h/2$ as a representative length in eq. (2), which leads to the estimate,

$$\Delta V \approx \frac{dQ/dr}{\pi\epsilon_0}\ln(h/a). \quad (3)$$

The corresponding capacitance per unit length along the antenna is,

$$\frac{dC}{dr} = \frac{\Delta V}{dQ/dr} \approx \frac{\pi\epsilon_0}{\ln(h/a)}, \quad (4)$$

and the total capacitance is,

$$C \approx \frac{\pi\epsilon_0 h}{\ln(h/a)}. \quad (5)$$

This estimate ignores the contribution to the capacitance of roughly $\pi\epsilon_0 a^2/d$ associated with the electric field in the gap d between the terminals of the antenna, as is reasonable when $d \approx a$ since then $\ln(h/a) \ll h/a \approx dh/a^2$.

2.1.2 Inductance

For a quick estimate of the self inductance L of the antenna, we note when the arms carry current I, the magnetic field $\mathbf{B}$ near the conductors varies with distance $r_\perp$ from as arm as,

$$B \approx \frac{\mu_0 I}{2\pi r_\perp}. \quad (6)$$

The magnetic flux $\Phi = \int \mathbf{B}\cdot d\mathbf{Area} = LI$ associated with the linear antenna is,

$$\Phi \approx Kh\int_a^h B\,dr_\perp \approx \frac{\mu_0 hI}{2\pi}\ln\frac{h}{a}, \quad (7)$$

where we note that the current drops from I to 0 over length h of each arm, and K is a constant of order 1. Then, our rough estimate of the self inductance L is,

$$L = \frac{\Phi}{I} \approx \frac{\mu_0 h}{2\pi}\ln\frac{h}{a}. \quad (8)$$

[2] In general the electric field is related to the scalar and vector potentials by $\mathbf{E} = -\boldsymbol{\nabla}V - \partial\mathbf{A}/\partial t = -\boldsymbol{\nabla}V - i\omega\mathbf{A}$, assuming a time dependence of the form $e^{i\omega t}$. Then, $\int_1^2 \mathbf{E}\cdot d\mathbf{l} = V_1 - V_2 - i\omega\int_1^2 \mathbf{A}\cdot d\mathbf{l}$. However, close to a small linear dipole antenna the electric field is much larger than the magnetic field (see, for example, [3]), and the contribution of the vector potential to the electric field in negligible in this region.

[3] The vanishing of the tangential component of the electric field along the (ideal) conductor implies that this conductor is an equipotential only if the vector potential can be neglected. For examples where this does not hold, see [4, 5].

2.1.3 Reactance

The reactance of a short linear antenna ($h \ll \lambda$) is largely due to its capacitance,

$$X_{\text{small linear}} = \omega L - \frac{1}{\omega C} \approx -\frac{1}{ckC} \approx -\frac{\ln(h/a)}{\pi\epsilon_0 ckh} = -\frac{Z_0}{\pi}\frac{\ln(h/a)}{kh} = -\frac{Z_0}{\pi^2}\frac{\lambda}{2h}\ln(h/a), \qquad (9)$$

where $\omega = kc = 2\pi c/\lambda$, $c = 1/\sqrt{\epsilon_0\,\mu_0}$ is the speed of light in vacuum, and,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0\, c = \frac{1}{\epsilon_0\, c} = 377\ \Omega. \qquad (10)$$

The reactance X of eq. (9) falls with increasing length h of the arms of the antenna, and vanishes when,

$$\omega = \frac{1}{\sqrt{LC}} \approx \sqrt{\frac{2\pi}{K\mu_0 h}\frac{1}{\pi\epsilon_0 h}} = \sqrt{\frac{2}{K}}\frac{c}{h} = kc = \frac{2\pi c}{\lambda}, \qquad i.e., \qquad h \approx \sqrt{\frac{2}{K}}\frac{\lambda}{2\pi}. \qquad (11)$$

A linear dipole antenna is known to have “resonance” ($X = 0$) when half-length $h \approx \lambda/4$. This tells us that $K \approx 8/\pi^2 = 0.81$ (and that our ”rough” estimates were rather good).

2.1.4 Relation between Reactance and “Free Oscillation”

As an aside, we note that frequencies at which the terminal reactance vanishes correspond to those of “free oscillation” of the antenna (with its terminals shorted).

In a “free oscillation”,[4] radiation is ignored and the (near) fields are standing waves that obey the Helmholtz equation, $(\nabla^2 + k^2)\psi = 0$, where ψ is any scalar component of the electric and magnetic fields. Electromagnetic energy is stored in the (near) fields, which oscillates between “electric” and “magnetic” terms, there being no exchange of energy with the perfect conductor.

For driven oscillations of a conductor, a nonzero terminal reactance implies an exchange of energy between the energy/voltage source and the (near) electromagnetic fields.

Thus, if the reactance is nonzero at some frequency, that frequency cannot correspond to a “free oscillation” (for which there is no exchange of energy between fields and conductors).

2.2 Small Loop Antenna

2.2.1 Inductance

One definition of a small loop antenna is that the spatial variation of the current around the loop can be neglected. In this case the self inductance L is essentially that of a circular loop/torus of, say, major radius b and minor radius a, supposing that all the current in on the surface because of the skin effect.

For a quick estimate we note when the loop carries current I the magnetic field near the conductor varies with distance $r_\perp$ roughly as,

$$B \approx \frac{\mu_0 I}{2\pi r_\perp}, \qquad (12)$$

[4] “Free oscillations” of (perfect) conductors were perhaps first discussed in [6]. See also, [7].

so the magnetic flux linked by the loop is,

$$\Phi = LI \approx 2\pi b \int_a^b B\, dr_\perp \approx \mu_0 b I \ln \frac{b}{a}\,, \tag{13}$$

and the self inductance L is,

$$L \approx \mu_0 b \ln \frac{b}{a} = \mu_0 b \left(\ln \frac{8b}{a} - 2.08 \right). \tag{14}$$

A more exact calculation using toroidal coordinates [8] shows that the number $2.08 = \ln 8$ in eq. (14) is actually 2 when $b \gg a$.

2.2.2 Capacitance

For a small loop antenna, the current is uniform around the loop, so there is no charge accumulation on the loop, and no associated capacitive reactance.

A loop antenna is driven at two terminals on the loop, with a small gap between them. A small capacitance, of order $\epsilon_0 a$ is associated with this gap,[5] which is usually neglected in discussions of the reactance of the loop, which then is essentially just its inductive reactance.[6]

As the size of the loop approaches a wavelength (or larger), the current, and hence also the surface charge density, varies around the ring, and there is an associated capacitive reactance. However, it is not obvious that this reactance is $1/i\omega C$ where C is the DC capacitance of a ring (with respect to "infinity").

Instead, we proceed by noting that a loop antenna is self resonant (reactance = 0) when its circumference $2\pi b$ is approximately equal to $n\lambda$ for integer n [11, 12]. For $n = 1$, we have that,

$$LC = \frac{1}{\omega^2} = \frac{\lambda^2}{4\pi^2 c^2} \approx \frac{b^2}{c^2} = \epsilon_0 \mu_0 b^2. \tag{15}$$

Then, using eq. (13) for the self inductance, we infer that the effective capacitance of the loop is,

$$C \approx \frac{\epsilon_0 \mu_0 b^2}{L} = \frac{\epsilon_0 b}{\ln \frac{b}{a}}. \tag{16}$$

This is larger than the DC capacitance by a factor of $4\pi^2$, according to the estimate of eq. (20) below.

DC Capacitance

For a quick estimate of the DC capacitance C we note when the loop supports charge Q, the electric field near the conductor varies with distance $r_\perp$ roughly as,

$$E \approx \frac{Q/2\pi b}{2\pi \epsilon_0 r_\perp}\,, \tag{17}$$

[5] See, for example, [10].

[6] A very different estimate is given in sec. 10-12 of [1], assuming that it is meaningful to consider the loop to be a capacitor consisting of two half loops; however, the current around a small loop is uniform, so there is no charge distribution around the loop, except at the terminals, and the estimate of [1] seems inappropriate.

so the electric field energy is,

$$U_e = \frac{Q^2}{2C} = \int \frac{\epsilon_2 E^2}{2}\, d\mathrm{Vol} \approx 2\pi b \int_a^b \frac{\epsilon_0 E^2}{2}\, r_\perp\, dr_\perp \approx \frac{Q^2}{2} \frac{1}{8\pi^3 \epsilon_0 b} \ln\frac{b}{a}\,, \tag{18}$$

and the capacitance C is,

$$C \approx \frac{8\pi^3 \epsilon_0 b}{\ln \frac{b}{a}}\,. \tag{19}$$

However, this estimate is not very accurate, and a better estimate of the DC capacitance is based on analysis in toroidal coordinates [9],

$$C \approx \frac{4\pi^2 \epsilon_0 b}{\ln \frac{8b}{a}} \approx \frac{4\pi^2 \epsilon_0 b}{\ln \frac{b}{a}} \approx \frac{4\pi^2 \epsilon_0 \mu_0 b^2}{L} = \frac{4\pi^2 b^2}{c^2 L}\,. \tag{20}$$

2.2.3 Reactance

The capacitive reactance of a small loop is, using eq. (16),

$$|X_C| = \frac{1}{\omega C} = \frac{L}{\epsilon_0 \mu_0 \omega b^2} = \frac{c^2 \omega L}{b^2 \omega^2} = \frac{4\pi^2 X_L \lambda^2}{b^2}\,, \tag{21}$$

which is much less that the inductive reactance $X_L = \omega L$. So, as previously noted, the capacitive reactance of small loop antennas is typically neglected.

The reactance of a small loop antenna is essentially that due to its self inductance,

$$X_{\text{small loop}} \approx \omega L \approx \mu_0 \omega b \ln\frac{b}{a} = \mu_0 c k b \ln\frac{b}{a} = Z_0 \frac{2\pi b}{\lambda} \ln\frac{b}{a} \gg Z_0. \tag{22}$$

A Appendix: Radiation Resistance of Small Antennas

For completeness, we include the well-known calculations of the radiation resistance $R_{\rm rad}$ of small antennas, noting that the time-average radiated power P is related to the peak current I_0 at the antenna terminals by,

$$P = \frac{I_0^2 R_{\rm rad}}{2} = \frac{\mu_0 \left|\ddot{p}\right|^2}{12\pi c} = \frac{\mu_0\, \omega^4 \left|p_0\right|^2}{12\pi c}\,, \qquad i.e., \qquad R_{\rm rad} = \frac{\mu_0\, \omega^4 \left|p_0\right|^2}{6\pi c I_0^2}\,, \tag{23}$$

where p_0 is the peak electric dipole moment of the antenna (or $p_0 = m_0$ in case the antenna has peak magnetic dipole moment m_0).

A.1 Short Linear Antenna

A short linear antenna of half length h along the z-axis has electric dipole moment p related to its linear charge density ρ by,

$$p = \int_{-h}^{h} \rho z\, a\, dz. \tag{24}$$

The charge density is related to the current distribution,

$$I(z,t) \approx I_0(1 - |z|/h)\, e^{i\omega t} \qquad (|z| < h), \tag{25}$$

by the continuity equation,

$$\dot{\rho} = -\frac{dI}{dz} \approx \pm\frac{I_0}{h}\, e^{i\omega t}, \tag{26}$$

so that,

$$\rho = \mp\frac{iI_0}{\omega h}\, e^{i\omega t} \qquad (|z| < h), \tag{27}$$

and,

$$p_0 = -\frac{iI_0 h}{\omega} \tag{28}$$

from eq. (24). Then, according to eq. (23) the radiation resistance is, recalling eq. (11),[7]

$$R_{\rm rad} = \frac{\mu_0\,\omega^4\,|p_0|^2}{6\pi c I_0^2} = \frac{\mu_0\,\omega^2 h^2}{6\pi c} = \frac{\pi\mu_0\, c\,(2h)^2}{6} = \frac{\pi Z_0}{6}\frac{(2h)^2}{\lambda^2} = 197\frac{(2h)^2}{\lambda^2}\ \Omega. \tag{29}$$

For an ("unmatched") small linear antenna with terminal impedance $Z \approx iX$ and reactance X given by eq. (9), the time-average radiated power when driven by a voltage source V_0 is, noting that $I_0 = |V_0/Z|$,

$$P_{\rm linear,unmatched} = \frac{V_0^2 R_{\rm rad}}{2\,|Z|^2} \approx \frac{V_0^2 R_{\rm rad}}{2X^2} \approx \frac{\pi^5 V_0^2}{12 Z_0 \ln^2(h/a)}\frac{(2h)^4}{\lambda^4}\,. \tag{30}$$

If the small linear antenna is "matched" to a line of (real) impedance $Z_{\rm line}$ ($\gg R_{\rm rad}$) then,

$$P_{\rm linear,matched} = \frac{V_0^2 R_{\rm rad}}{2Z_{\rm line}^2} \approx \frac{\pi V_0^2 Z_0}{12 Z_{\rm line}^2}\frac{(2h)^2}{\lambda^2}\,. \tag{31}$$

A.2 Small Loop Antenna

A small loop antenna (of radius b) has azimuthally symmetric current $I(\phi,t) = I_0\, e^{-i\omega t}$, such that the peak magnetic dipole moment is,

$$m_0 = \pi b^2 I_0, \tag{32}$$

and radiation resistance,

$$R_{\rm rad} = \frac{\mu_0 \omega^4\,|m_0|^2}{6\pi c^3 I_0^2} = \frac{\pi\mu_0\omega^4 b^4}{6c^3} = \frac{\pi\mu_0 c}{6}\frac{(2\pi b)^4}{\lambda^4} = \frac{\pi Z_0}{6}\frac{(2\pi b)^4}{\lambda^4}\,. \tag{33}$$

[7] The approximation (25) is not very accurate for "resonance" with $h \approx \lambda/4$, for which eq. (29) gives $R_{\rm rad,resonance} \approx 197/4 = 49\ \Omega$. At "resonance", $I(z,t) \approx I_0 \cos kz\, e^{i\omega t}$, so $\dot{\rho} \approx kI_0 \sin kz\ e^{i\omega t}$, $\rho \approx -iI_0 \sin kz\ e^{i\omega t}/c$, $p_0 \approx -2I_0 c/\omega^2$, $R_{\rm rad} \approx 2\mu_0 c/3\pi = 2Z_0/3\pi = 80\ \Omega$, which is closer to the actual value of 71 Ω at "resonance".

For an ("unmatched") small loop antenna with reactance X given by eq. (22), the time-average radiated power when driven by a voltage source V_0 is,

$$P_{\rm loop,unmatched} \approx \frac{V_0^2 R_{\rm rad}}{2X^2} \approx \frac{\pi V_0^2}{12 Z_0 \ln^2(b/a)} \frac{(2\pi b)^2}{\lambda^2}\,. \tag{34}$$

If the small loop antenna is "matched" to a line of impedance $Z_{\rm line}$ then,

$$P_{\rm loop,matched} = \frac{V_0^2 R_{\rm rad}}{2Z_{\rm line}^2} \approx \frac{\pi V_0^2 Z_0}{12 Z_{\rm line}^2} \frac{(2\pi b)^4}{\lambda^4}\,. \tag{35}$$

Thus, a "matched", small loop antenna of circumference $2\pi b$ radiates much less power than a "matched", small linear antenna of total length $2h = 2\pi b$.[8]

B Comments on the Wave Speed

In the field theory of electromagnetism, electromagnetic waves propagate in the fields, which in the case of good conductors are nonzero mainly outside the conductors. In this view, it a "natural" that the speed of electromagnetic waves is that associated with the medium surrounding the conductors, which differs little from vacuum in most antenna applications. Then, the speed of waves in (or better, on the surface of) the conductors of an antenna is close to the speed of light in vacuum.

We illustrate this for *self-resonant* linear and loop antennas, which have zero reactance by definition. For linear antennas, the smallest self-resonant antennas have arms of length $h \approx \lambda/4$, while for loop antennas the circumference $2\pi b$ is approximately the wavelength λ. For the linear self-resonant antenna, the current standing wave has the form $\cos kz \cos \omega t = (1/2([cos(kz - \omega t) + \cos)kz + \omega t)]$, where the wave number $k = 2\pi/h$ is very close to $2\pi/\lambda = c/\omega$, so the speed of the current wave is $\omega/k \approx c$. Similarly, the traveling wave of current on the self-resonant loop has the form $\cos(ks \pm \omega t)$, where s is the arc length around the loop of radius $b \approx \lambda$, so $k = 2\pi/b \approx c/\omega$ and again the wave speed is $\omega/k \approx c$.

On the other hand, an antenna can be thought of as the final element in the transmission line from the power source. In the circuit theory of transmission lines (due to Heaviside (1876) [13], the speed of waves along a two-conductor transmission line is,

$$v = 1/\sqrt{\tilde{L}\tilde{C}}\,, \tag{36}$$

where $\tilde{L}$ and $\tilde{C}$ are the capacitance per unit length along the line.

If we used the estimates (5) and (8) for the capacitance and inductance of a linear antenna with arms of length h, the transmission-line formula (36) would suggest that the wave speed of the current on each arm is $v_{\rm linear} = 1/\sqrt{LC/h^2} = 1/\sqrt{\epsilon_0\mu_0/2} = \sqrt{2}\,c$, while use of the estimates (14) and (16) for a loop of circumference $2\pi b$ would suggest that the wave speed is $v_{\rm loop} = 1/\sqrt{LC/(2\pi b)^2} = 2\pi\,c$.

These inconsistencies reflect that the transmission-line formula (36) holds only in the limit of a very long line of two parallel conductors. It is impressive that the earliest consideration

[8] An "unmatched", small loop antenna of circumference $2\pi b$ radiates more power than an "unmatched", small linear antenna of total length $2h = 2\pi b$ provided $2\pi b \lesssim \lambda/10$, but the radiated power is quite small.

of waves on conductors, by Kirchhoff in 1857 [14], via consideration of short wire segments in the action-at-a-distance theory of Weber [15], deduced that the wave speed was c.[9]

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