Electromagnetic Momentum of a Capacitor in a Uniform Magnetic Field

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(June 18, 2006; updated October 9, 2024)

1 Problem

Analytic calculations of the electric field of a parallel-plate capacitor are notoriously difficult [1]-[13]. The usual goal of such efforts is a calculation of the correction to the hard-edge approximation, $C = A/4\pi d$ (in Gaussian units), of the capacitance for plates of area A separated by distance d.

Calculate instead the electromagnetic momentum of the parallel-plate capacitor if it resides in a uniform magnetic field that is parallel to the capacitor plates.

Consider also the case of a capacitor whose electrodes are caps of polar angle $\theta_0 < \pi/2$ on a sphere of radius a.

In both cases, the remaining space is vacuum.

2 Solution

The principal result of this exercise is that the electromagnetic momentum of a parallel-plate capacitor of central field $\mathbf{E}_0$ in a uniform magnetic field $\mathbf{B}_0$ due to a long solenoid is only 1/2 of the naive estimate of $\mathbf{E}_0 \times \mathbf{B}_0$ times (Volume/$4\pi c$).[1] Furthermore, if the electromagnetic momentum is interpreted as being stored in the electromagnetic fields, then only 2/3 of the electromagnetic momentum is localized near the capacitor, while the remaining 1/3 is localized near the coils of the magnet, if there is no shielding of the fringe field of the capacitor.

2.1 Electromagnetic Momentum

For systems in which effects of radiation and of retardation can be ignored, the electromagnetic momentum can be calculated in various equivalent ways [15, 16],[2]

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{\Phi \mathbf{J}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\, d{\rm Vol}, \tag{1}$$

[1] A previous discussion [14] of the electromagnetic momentum of a capacitor in an electric field missed this factor of 1/2.

[2] The third form of eq. (1) indicates that if the magnetic field is created by steady currents in a good/super conductor, over which the scalar potential Φ is constant, then $\mathbf{P}_{\rm EM} = (\Phi/c^2)\int \mathbf{J}\, d{\rm Vol} = 0$. Similarly, currents in a resistive conductor are associated with scalar potential that is maintained by a "battery," independent of other external fields, which latter don't affect the field momentum. Hence, we restrict our attention to magnetic fields created by currents that are not shielded from external fields, such as a rotating spherical shell of charge.

where ϱ is the electric charge density, $\mathbf{A}$ is the magnetic vector potential (in the Coulomb gauge where $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, Φ is the electric (scalar) potential (also in the Coulomb gauge), $\mathbf{J}$ is the electric current density, and c is the speed of light. The first form is due to Faraday[3] and Maxwell [21], the second form is due to Poynting [22], J.J. Thomson [23, 24, 25] and Poincaré [26], the third form was introduced by Furry [27], and the fourth form is due to Aharonov [28].

The four forms of eq. (1) lead to ambiguities of interpretation as to where the electromagnetic momentum is located, since the four integrands cannot all be the same physical momentum density. The author takes the "Maxwellian" view that the electromagnetic field momentum is stored in the electromagnetic fields, with density $\mathbf{p}_{\rm EM} = \mathbf{E} \times \mathbf{B}/4\pi c$ (although Maxwell himself was only aware of the first form of eq. (1), and associated the term $\varrho \mathbf{A}/c$ with the density of electromagnetic momentum). This ambiguity suggests to some that the electromagnetic momentum does not have a clear meaning when associated with charges and currents, in contrast to the case of free fields where it is the classical precursor of the momentum of quantum photons [29]. Nonetheless, consistency between mechanics and electromagnetism is only achieved if an electromagnetic momentum can be associated with charges, currents and fields. Other examples by the author that illustrate this point include [30]-[36].

The existence of four equivalent methods of calculation of the electromagnetic momentum permits us to choose whichever form is most convenient in a particular situation. However, it also provides some general guidance as to the sensitivity of these calculations to details at large distances. For example, if the electric charge distribution ϱ is spatially localized then the first form of eq. (1) permits a calculation using only knowledge of the vector potential in that localized region. This might suggest that details of the electric and magnetic fields at large distances will be unimportant if we use the second form of eq. (1) to calculate the electromagnetic momentum. However, the third form eq. (1) requires detailed knowledge of the electric potential at the location of the currents that generate the magnetic field, which may be far from the electric charges. This warns us that detailed knowledge of the electric field far from the charges is in general needed when using the second form of eq. (1).

Another perspective is that the electromagnetic momentum (1) is a (bi)linear function of the electric and magnetic fields, so its value is more sensitive to the behavior of the fields at large distances than, say, a calculation of field energy which is quadratic in the electric and magnetic field strengths. This point is illustrated in sec. 2.3 by a possibly surprising result for the electromagnetic momentum of a capacitor in a uniform magnetic field.

2.2 Electric Dipole $\mathbf{p} = q\mathbf{d}$

We first consider the case of an electric dipole $\mathbf{p} = q\mathbf{d}$ that consists of point charges $\pm q$ at positions $\mathbf{d}^+$ and $\mathbf{d}^-$ where $\mathbf{d} = \mathbf{d}^+ - \mathbf{d}^-$.

The uniform magnetic field of strength $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$ coud be created by a (long) solenoid (rotating cylindrical shell of charge) whose axis of symmetry is the z axis. In the vicinity of

[3]Electromagnetic momentum can be identified with the *electro-tonic* state, first discussed by Faraday in Art. 60 of [17]. Other mentions by Faraday of the electrotonic state include Art. 1661 of [18], Arts. 1729 and 1733 of [19], and Art. 3269 of [20].

the capacitor the vector potential of the magnetic field is azimuthal, with value $A_\phi = \rho B_0/2$,[4] so that we can write,

$$\mathbf{A} = \frac{\varrho}{2} B_0\, \hat{\boldsymbol{\phi}} = \frac{B_0}{2}(-\rho \sin\phi\, \hat{\mathbf{x}} + \rho\cos\phi\, \hat{\mathbf{y}}) = \frac{B_0}{2}(-y\, \hat{\mathbf{x}} + x\, \hat{\mathbf{y}}) = \frac{B_0}{2}\, \mathbf{z} \times \hat{\mathbf{r}} = \frac{\mathbf{B}_0 \times \mathbf{r}}{2}\,, \tag{2}$$

at a point $\mathbf{r} = (x, y, z) = (\rho, \phi, z)$ in a cylindrical coordinate system.

It is most straightforward in the present case to evaluate the electromagnetic momentum using the first form of eq. (1). Then,

$$\mathbf{P}_{\mathrm{EM},1} = \int \frac{\varrho \mathbf{A}}{c}\, d\mathrm{Vol} = \frac{q\mathbf{B}_0}{2c} \times (\mathbf{d}^+ - \mathbf{d}^-) = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}\,. \tag{3}$$

Since the electromagnetic momentum (1) is a linear function of the charge distribution ϱ (as well as a linear function of the electric field $\mathbf{E}$ and the scalar potential Φ), the electromagnetic momentum of an extended charge distribution in a uniform magnetic field can be calculated by linear superposition. In particular, **the electromagnetic momentum for any charge distribution with cylindrical symmetry (entirely within a uniform magnetic field $\mathbf{B}_0$ for which the interior vector potential is given by eq. (2)) that is a superposition of electric dipoles obeys eq. (3), where $\mathbf{p}$ is the total electric dipole moment of the charge distribution.**[5,6] This result is confirmed for special cases in secs. 2.3-4 and 2.6 below.[7] Some examples that do not obey eq. (3) are given in [39].

Additional calculations of the electromagnetic momentum of an electric dipole are given in the Appendix.[8]

---

[4]This vector potential also holds inside a rotating spherical shell of charge, as discussed in Prob. 12(a) of [37].

[5]The result (3) can be inferred from Table II of [38]. If the sources of the external magnetic field were magnetic monopoles rather than electric currents, the electromagnetic field momentum of an electric dipole in the magnetic field would be zero, as can be inferred from Table I of [38]. This fact is desirable in that for the latter case the system would contain no "moving parts" with mechanical momentum, so there could be no "hidden mechanical momentum" to compensate for a nonzero electromagnetic field momentum of a system whose center of mass/energy is at rest.

[6]It is noted in [39] that the case of a "point" magnetic dipole $\mathbf{m}$ and a "point" electric dipole $\mathbf{p}$ does not obey eq. (3), because the magnetic field is slightly nonuniform over even a "point" electric dipole.

[7]See also, [40].

[8]Equation (3) was deduced with the tacit assumption that the electric dipole is inside the uniform magnetic field $\mathbf{B}_0$. Suppose instead that the dipole is outside the solenoid (of radius a), where the magnetic field is very small. The vector potential at the dipole then has the form,

$$\mathbf{A} = \frac{a^2}{2\rho} B_0\, \hat{\boldsymbol{\phi}} = \frac{a^2}{2\rho^2}\rho B_0\, \hat{\boldsymbol{\phi}} = \frac{a^2 B_0}{2\rho^2}\, \mathbf{z} \times \hat{\mathbf{r}} = \frac{a^2}{\rho^2}\frac{\mathbf{B}_0 \times \mathbf{r}}{2}\,, \tag{4}$$

and,

$$\mathbf{P}_{\mathrm{EM},1} = \int \frac{\varrho \mathbf{A}}{c}\, d\mathrm{Vol} = \frac{qa^2\mathbf{B}_0}{2c} \times \left(\frac{\mathbf{d}^+}{\rho^{+2}} - \frac{\mathbf{d}^-}{\rho^{-2}}\right) \approx \frac{a^2 \mathbf{B}_0 \times \mathbf{p}}{2c\rho^2}\,. \tag{5}$$

In eq. (3), $\mathbf{B}_0$ happens to be the field at the location of the dipole, while in eq. (5) $\mathbf{B}_0$ is the magnetic field in the distant solenoid and not the (near-zero) magnetic field at the dipole.

Equation (5) indicates that the field momentum of an electric dipole in a magnetic field can take on almost any value. For example, if the "uniform" magnetic field at the electric dipole $\mathbf{p}$ is due to a distant magnetic dipole $\mathbf{m}$, then the field momentum of the system is given by $\mathbf{P}_{\mathbf{EM}} = \mathbf{E} \times \mathbf{m}/c$ (as first deduced

2.3 Parallel-Plate Capacitor

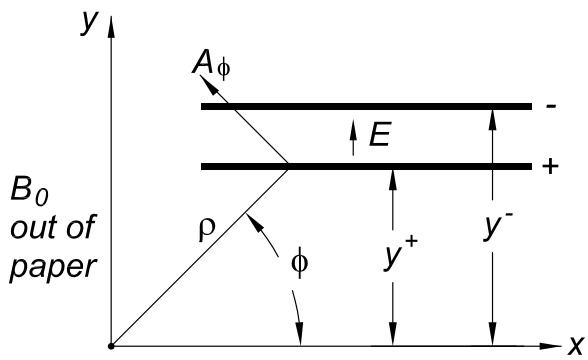
We next consider a parallel-plate capacitor that has plates of area A and separation d that are parallel to the x-z plane and at distances y^+ and y^- from it, where $d = y^- - y^+$, as shown in the figure below.

The uniform magnetic field of strength B_0 is again created by a (long) solenoid whose axis of symmetry is the z axis, so its vector potential is again given by eq. (2).

If the two parallel plates of the capacitor have the same shape, then the surface charge distributions on the two plates have the same form, $\sigma^+ = -\sigma^- \equiv \sigma$, and the total charge distribution is a superposition of electric dipoles parallel to the y axis. In this case we can evaluate the electromagnetic momentum using eq. (3) (valid for the capacitor inside the magnetic field),

$$\mathbf{P}_{\mathrm{EM},1} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c} = \frac{B_0\,\hat{\mathbf{z}} \times -Qd\,\hat{\mathbf{y}}}{2c} = \frac{QdB_0}{2c}\,\hat{\mathbf{x}} = \frac{E_0 B_0}{8\pi c}\,\mathrm{Vol}\,\hat{\mathbf{x}}\,, \tag{8}$$

where $\mathbf{p} = -Qd\,\hat{\mathbf{y}}$ is the total electric dipole moment, $Q = \int \sigma\; d\mathrm{Area}$ is the charge on each plate of the capacitor, and $E_0 = 4\pi Q/A$ is the electric field in the capacitor neglecting edge effects.



We could also use the first form of eq. (1), together with eq. (2), to find,

$$\mathbf{P}_{\mathrm{EM},1} \;=\; \int \frac{\varrho\mathbf{A}}{c}\; d\mathrm{Vol} = \frac{1}{c}\left[\int_+ \sigma^+\mathbf{A}^+\; d\mathrm{Area}^+ + \int_- \sigma^-\mathbf{A}^-\; d\mathrm{Area}^-\right]$$

by J.J. Thomson [25, 41, 42, 43]). Then,

$$\mathbf{P}_{\mathrm{EM}} = \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} \times \frac{\mathbf{m}}{c} = \frac{\mathbf{B}\times\mathbf{p}}{c} + \frac{3\hat{\mathbf{r}} \times [(\mathbf{p}\cdot\hat{\mathbf{r}})\mathbf{m} - (\mathbf{m}\cdot\hat{\mathbf{r}})\mathbf{p}]}{cr^3} + 2\frac{\mathbf{m}\times\mathbf{p}}{cr^3}\,. \tag{6}$$

The field momentum (6) equals $\mathbf{B}\times\mathbf{p}/c$ if $\mathbf{m} \parallel \mathbf{p}$, but is twice this if $\mathbf{p} \parallel \hat{\mathbf{r}} \perp \mathbf{m}$, and half this if $\mathbf{p} \perp \hat{\mathbf{r}} \parallel \mathbf{m}$, and the negative of this if $\mathbf{m}$, $\mathbf{p}$ and $\hat{\mathbf{r}}$ are mutually orthogonal.

Further, a general argument has been given by B.Y.-K. Hu [39], who extended eq. (65) of [38] to find,

$$\mathbf{P}_{\mathrm{EM},1} = (\mathbf{p}\cdot\boldsymbol{\nabla})\frac{\mathbf{A}}{c} = \frac{\mathbf{B}\times\mathbf{p}}{c} + \boldsymbol{\nabla}\left(\frac{\mathbf{p}\cdot\mathbf{A}}{c}\right) = \frac{\mathbf{B}\times\mathbf{p}}{c} - \int (\mathbf{p}\cdot\mathbf{J})\frac{\mathbf{r}-\mathbf{r}_p}{c^2\left|\mathbf{r}-\mathbf{r}_p\right|^3}\,d\mathrm{Vol}, \tag{7}$$

where $\mathbf{B}$ is the magnetic field at the dipole and $\mathbf{r}$ points from the current element to the dipole. The integral in eq (7) is nonzero in general, and can have almost any value for suitably complex current distributions.

Thus, the result (3) is a special case in the mathematical sense, although the assumption there of circular symmetry for the magnetic field is "practical."

$$
\begin{aligned}
&= \frac{B_0}{2c}\left[\int \sigma(-y^+\,\hat{\mathbf{x}} + x^+\,\hat{\mathbf{y}})\; d\mathrm{Area} - \int \sigma(-y^-\,\hat{\mathbf{x}} + x^-\,\hat{\mathbf{y}})\; d\mathrm{Area}\right] \\
&= \frac{B_0}{2c}(y^- - y^+)\int \sigma\; d\mathrm{Area}\,\hat{\mathbf{x}} = \frac{B_0 Q d}{2c}\,\hat{\mathbf{x}} = \frac{B_0 p}{2c}\,\hat{\mathbf{x}} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}\,. \qquad (9)
\end{aligned}
$$

The result (8) holds no matter what the shape of the capacitor plates, and they are valid for any separation d. This result is remarkable in that it is exactly 1/2 of the naive expectation based on form 2 of eq. (1),

$$
\mathbf{P}_{\mathrm{EM},2} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\; d\mathrm{Vol} \overset{?}{\approx} \frac{E_0 B_0}{4\pi c}\;\mathrm{Vol}\,\hat{\mathbf{x}}. \qquad (10)
$$

In the interpretation that the electromagnetic momentum is stored in the electromagnetic fields, we infer from eqs. (8) and (9) (which assume a uniform magnetic field with cylindrical symmetry) that outside the capacitor, in its fringe field, the stored momentum is approximately $-1/2$ that stored within the nominal volume.[9] This result is surprising in that we can typically neglect the region outside the capacitor in considerations of capacitance and stored energy. However, energy is quadratic while momentum is linear in the electric field strength, so that field momentum is much more sensitive to fringe-field effects than is field energy.[10]

2.4 Spherical Capacitor inside a Spherical Magnet

In this section we consider a capacitor whose electrodes are conducting spherical caps on a sphere of radius a. The caps extend over polar angle $0 \le \theta \le \theta_0 < \pi/2$ in a spherical coordinate system whose z axis is the symmetry axis of the capacitor.[11]

[9] In the following section we deduce that 1/3 of the total electromagnetic momentum is located far from the capacitor, so that the momentum in the fringe field near a spherical capacitor is actually $-2/3$ that stored in the nominal volume.

[10] The forms (8)-(9) depend on the assumption of cylindrical symmetry for the magnetic field $\mathbf{B}_0$ and the (possibly distant) currents that produce it. An extreme alternative (suggested by B.Y.-K. Hu [39]) is for the uniform magnetic field to be produced by sheets of current flow in the $\mp x'$ directions in planes at $y' = \pm b$, in (x', y') coordinates rotated by angle θ with respect to the x-axis, for which the vector potential between these sheets is $A_{x'} = B_0 y'$ and $A_{y'} = 0$; *i.e.*, $\mathbf{A} = B_0(-x\sin\theta\cos\theta + y\cos^2\theta)\,\hat{\mathbf{x}} + B_0(-x\sin^2\theta + y\sin\theta\cos\theta)\,\hat{\mathbf{y}}$, and $\mathbf{A} = 0 = \mathbf{B}$ outside the sheets. Then, if the capacitor is inside the magnetic field,

$$
\begin{aligned}
\mathbf{P}_{\mathrm{EM},1} &= \int \frac{\varrho\mathbf{A}}{c}\; d\mathrm{Vol} = \frac{B_0}{c}\{[Q(-x^+\sin\theta\cos\theta + y^+\cos^2\theta) + (-Q)(-x^-\sin\theta\cos\theta + y^-\cos^2\theta)]\,\hat{\mathbf{x}} \\
&\qquad + [Q(-x^+\sin^2\theta + y^+\sin\theta\cos\theta) + (-Q)(-x^-\sin^2\theta + y^-\sin\theta\cos\theta)]\,\hat{\mathbf{y}}\} \\
&= \frac{B_0 p}{c}(\cos^2\theta\,\hat{\mathbf{x}} + \sin\theta\cos\theta\,\hat{\mathbf{y}}) = \cos^2\theta\frac{\mathbf{B}_0\times\mathbf{p}}{c} + \sin\theta\cos\theta\frac{B_0 p}{c}\,\hat{\mathbf{y}}, \qquad (11)
\end{aligned}
$$

while the field momentum is zero if the capacitor is outside the magnetic field. Note that the field momentum is proportional to $\mathbf{B}_0 \times \mathbf{p}/c$ only for the special cases of $\theta = 0$ or $90°$.

Since the field momentum within the nominal volume of the capacitor is approximately $\mathbf{B}_0 \times \mathbf{p}/c$, we learn that the field momentum outside this nominal volume, but between the two current sheets, which corresponds to the integral term in eq. (7), is $-\sin^2\theta\mathbf{B}_0\times\mathbf{p}/c + \sin\theta\cos\theta(B_0 p/c)\,\hat{\mathbf{y}}$.

[11] The case of a sphere with a fixed surface charge distribution that varies as $\cos\theta$ was considered by Romer [29], which first interested the author in this type of problem.

We suppose that the uniform external magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{y}}$ is along the y-axis. Then, the electric potential Φ has the form,

$$\Phi(r,\theta) = \begin{cases} \sum_{n\ \mathrm{odd}} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) & (r<a), \\ \sum_{n\ \mathrm{odd}} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) & (r>a). \end{cases} \tag{12}$$

It is not easy to determine the Fourier coefficients A_n for the general case of spherical caps of angle θ_0, but it turns out that we need to know only the coefficient A_1, which is related to the dipole moment of the capacitor. Of course, the Fourier coefficients are nonzero only for odd n, since the charge distributions on the two plates are equal and opposite.

We are interested in the x-component of the electromagnetic momentum, which we calculate using the second form of eq. (1),

$$P_{\mathrm{EM},2,x} = -\int \frac{E_z B_0}{4\pi c}\, d\mathrm{Vol}. \tag{13}$$

The z-component of the electric field is given by,

$$\begin{aligned} E_z &= \cos\theta E_r - \sin\theta E_\theta = -P_1(\cos\theta)\frac{\partial\Phi}{\partial r} + \frac{\sin\theta}{r}\frac{\partial\Phi}{\partial\theta} \\ &= \begin{cases} -\sum_{n\ \mathrm{odd}} A_n \frac{r^{n-1}}{a^n}\left(nP_1P_n + \sin^2\theta P_n'\right) & (r<a), \\ \sum_{n\ \mathrm{odd}} A_n \frac{a^{n+1}}{r^{n+2}}\left((n+1)P_1P_n - \sin^2\theta P_n'\right) & (r>a), \end{cases} \end{aligned} \tag{14}$$

noting that $dP_n(\cos\theta)/d\theta = -\sin\theta\, dP_n(\cos\theta)/d\cos\theta = -\sin\theta P_n'$.

We first consider the volume integral for the region $r>a$. In particular, if the magnetic field is uniform for all $r>a$, then this integral includes the factor,

$$\begin{aligned} &\int_{-1}^{1} d\cos\theta\, \left((n+1)P_1P_n - \sin^2\theta P_n'\right) = \frac{4}{3}\delta_{1n} - \int_{-1}^{1} d\mu\,(1-\mu^2)P_n'(\mu) \\ =\ & \frac{4}{3}\delta_{1n} - P_n|_{-1}^{1} + \mu^2 P_n|_{-1}^{1} - \int_{-1}^{1} d\mu\, 2\mu P_n(\mu) = \frac{4}{3}\delta_{1n} - \frac{4}{3}\delta_{1n} = 0. \end{aligned} \tag{15}$$

Thus, the contribution to the electromagnetic momentum for $r>a$ vanishes at every order n, and so it appears that the total electromagnetic momentum stored in this region is zero.

For the region $r<a$ the θ integral is,

$$\begin{aligned} &\int_{-1}^{1} d\cos\theta\, \left(nP_1P_n + \sin^2\theta P_n'\right) = \frac{2}{3}\delta_{1n} + \int_{-1}^{1} d\mu\,(1-\mu^2)P_n'(\mu) \\ =\ & \frac{2}{3}\delta_{1n} + P_n|_{-1}^{1} - \mu^2 P_n|_{-1}^{1} + \int_{-1}^{1} d\mu\, 2\mu P_n(\mu) = \frac{2}{3}\delta_{1n} + \frac{4}{3}\delta_{1n} = 2\delta_{1n}. \end{aligned} \tag{16}$$

Thus, only the $n=1$ term of the potential for $r<a$ contributes to the electromagnetic momentum. We can identify the coefficient A_1 as p/a^2, where p is the dipole moment of the charged capacitor (according to an observer at $r>a$). The electric field $E_{z,1} = -p/a^3$ due to the $n=1$ term is constant within the sphere of radius a.

The electromagnetic momentum according to the second form of eq. (1) is then,

$$P_{\mathrm{EM},2,x} = -\int \frac{E_z B_0}{4\pi c}\, d\mathrm{Vol} = -\frac{E_{z,1} B_0}{4\pi c}\ \mathrm{Vol} = \frac{B_0 p}{3c}\,, \tag{17}$$

for any angle θ_0 of the spherical capacitor.

This simplicity of this result is gratifying, but it disagrees with the previous result (3) that the electromagnetic momentum of an electric dipole at right angles to a uniform magnetic field is $B_0 p/2c$.

The defect of the result (17) is that it is based on the assumption that the magnetic field is uniform at large distances from the capacitor. But, any real magnetic field, whose source currents lie within a bounded volume, falls to zero at very large distances. And, as we noted at the end of sec. 2.1, the details of the fields at large distance are important when using form 2 of eq. (1) to calculate the electromagnetic momentum.

A model for a magnetic field that is uniform near the origin and well defined at large distances is a sphere of (large) radius $b > a$ on which there exists a surface-current density that varies as $\mathbf{K} = 3c\mathbf{B}_0 \times \hat{\mathbf{r}}/8\pi$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the sphere. That is, the surface-current density varies as $\sin\theta$. The magnetic field is uniform within the sphere, while outside the sphere the field is that of the magnetic dipole $\mathbf{m} = b^3\mathbf{B}_0/2 = b^3 B_0\,\hat{\mathbf{y}}/2$ located at the center of the sphere. See also sec. 3.3 below.

The calculation of the electromagnetic momentum via form 2 of eq. (1) for the spherical capacitor plus spherical magnet is the same as that given above for $r < b$, where the magnetic field is uniform.

Now, we must calculate the electromagnetic momentum in the region $r > b$, where the magnetic field has the dipole form,

$$\begin{aligned} \mathbf{B}(r > b) &= \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \\ &= \frac{m}{r^3}[3\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{x}} + (3\sin^2\theta\sin^2\phi - 1)\,\hat{\mathbf{y}} + 3\sin\theta\cos\theta\sin\phi\,\hat{\mathbf{z}}], \end{aligned} \tag{18}$$

since,

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}. \tag{19}$$

The electric field of the spherical capacitor has rectangular components,

$$\begin{aligned} \mathbf{E} &= E_r\,\hat{\mathbf{r}} + E_\theta\,\hat{\boldsymbol{\theta}} \qquad (20) \\ &= (E_r\sin\theta + E_\theta\cos\theta)\cos\phi\,\hat{\mathbf{x}} + (E_r\sin\theta + E_\theta\cos\theta)\sin\phi\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}}, \end{aligned}$$

noting that,

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}. \tag{21}$$

The cross product is,

$$\begin{aligned} \mathbf{E} \times \mathbf{B}(r > b) \qquad (22) \\ = \frac{m}{r^3}[3(E_r\sin\theta + E_\theta\cos\theta)\sin\theta\cos\theta\sin^2\phi - E_z(3\sin^2\theta\sin^2\phi - 1)]\,\hat{\mathbf{x}} \\ -\frac{3m}{r^3}E_\theta\sin\theta\sin\phi\cos\phi\,\hat{\mathbf{y}} - \frac{m}{r^3}(E_r\sin\theta + E_\theta\cos\theta)\cos\phi\,\hat{\mathbf{z}}. \end{aligned}$$

On performing the azimuthal part of the volume integral, the only surviving terms are,

$$\begin{aligned}\int_0^{2\pi} \mathbf{E}\times\mathbf{B}(r>b)\,d\phi &= \frac{\pi m}{r^3}[3(E_r\sin\theta+E_\theta\cos\theta)\sin\theta\cos\theta - E_z(3\sin^2\theta-2)]\,\hat{\mathbf{x}} \\ &= \frac{\pi m}{r^3}\sum_{n\ \mathrm{odd}} A_n\frac{a^{n+1}}{r^{n+2}}\left(2(n+1)\mu P_n+(1-\mu^2)P_n'\right)\,\hat{\mathbf{x}} \qquad (23)\end{aligned}$$

recalling eq. (14), and where $\mu=\cos\theta=P_1$. The polar part of the volume integral includes the factor,

$$\int_{-1}^{1} d\mu\,\left(2(n+1)\mu P_n+(1-\mu^2)P_n'\right) = \int_{-1}^{1} d\mu\,\left(2(n+1)\mu P_n+2\mu P_n\right) = 4\delta_{1n}. \qquad (24)$$

The contribution to the electromagnetic momentum at $r>b$ is thus,

$$\mathbf{P}_{\mathrm{EM},2}(r>b) = \int_{r>b}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} = \frac{a^2A_1m}{c}\int_b^\infty\frac{r^2\,dr}{r^6}\,\hat{\mathbf{x}} = \frac{a^2A_1m}{3b^3c}\,\hat{\mathbf{x}} = \frac{B_0p}{6c}\,\hat{\mathbf{x}}. \qquad (25)$$

Combining this with eq. (17), which gives the electromagnetic momentum for $r<b$, we find the total electromagnetic momentum of a spherical capacitor inside a $\sin\theta$ spherical magnet to be,

$$\mathbf{P}_{\mathrm{EM},2} = \frac{B_0p}{2c}\,\hat{\mathbf{x}} = \frac{\mathbf{B}_0\times\mathbf{p}}{2c}\,, \qquad (26)$$

which now agrees with eq. (3).

It is noteworthy that 1/3 of the electromagnetic momentum is located outside the coil of the spherical magnet, according to form 2 of eq. (1), no matter how large is the magnet compared to the capacitor inside it. This result holds for any two-electrode capacitor with midplane symmetry, since its potential obeys eq. (12) for $r>a$ where a is the radius of a sphere that completely enclosed the capacitor.

Thus, we find the result (9) of sec. 2.3 to be even more impressive as we now understand that the electromagnetic momentum located close to the capacitor sums to only 1/3 of the naive estimate $\mathbf{E}_0\times\mathbf{B}_0\,\mathrm{Vol}/4\pi c$.

For completeness, we also calculate the electromagnetic momentum using the third form of eq. (1),

$$\begin{aligned}\mathbf{P}_{\mathrm{EM},3} &= \int_{r=b}\frac{\Phi(b)\mathbf{K}}{c^2}\,d\mathrm{Area} = \frac{3\mathbf{B}_0}{8\pi c}\times\sum_{n\ \mathrm{odd}} A_n\left(\frac{a}{b}\right)^{n+1}\int_{-1}^{1}2\pi b^2\,d\cos\theta\;P_n(\cos\theta)\,\hat{\mathbf{r}} \\ &= \frac{3b^2\mathbf{B}_0}{4c}\times\sum_{n\ \mathrm{odd}} A_n\left(\frac{a}{b}\right)^{n+1}\int_{-1}^{1}d\cos\theta\;P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}}+\sin\theta\,\hat{\boldsymbol{\rho}}) \\ &= \frac{\mathbf{B}_0\times a^2A_1\,\hat{\mathbf{z}}}{2c} = \frac{\mathbf{B}_0\times\mathbf{p}}{2c}\,, \qquad (27)\end{aligned}$$

in agreement with eq. (3).

Thus, we have an explicit example of a capacitor in a magnetic field for which the electromagnetic momentum is calculated to have the same value using the first three forms of eq. (1).

2.5 Hidden Mechanical Momentum

The electromagnetic momentum (1) can be nonzero for configurations of static charge distributions combined with steady electric currents. From a mechanical point of view, such systems are at rest, so it is counterintuitive that they can contain a nonzero momentum.

In footnote 2 above we saw that if the currents flow in perfect conductors, which are equipotentials, the total electromagnetic momentum of the system of currents plus fixed charges would be zero. Hence, in the examples above we have tacitly assumed that the currents do not flow in superconductors.

An idealized model for the surface currents $\mathbf{K}$ of the preceding sections is that they are due to a nonconducting tubes at rest that contain a circulating incompressible liquid of charged molecules, and that adjacent tubes have oppositely charged molecules whose flow has opposite senses of rotation. In this model the only matter in motion is the charge carriers, and the structure is electrically neutral. The electric potential on the incompressible charged fluid is that due to the fixed charges elsewhere in the system.

In this case, as first noted by Shockley [44], and more explicity by Coleman and Van Vleck [45] and by Furry [27], as the charges move through a spatially varying electric potential Φ their relativistic mass changes according to $-e\Phi/c^2$ so that the (quasistatic) system possesses a "hidden" mechanical momentum,

$$\mathbf{P}_{\text{mech,hidden}} = -\int \frac{\Phi \mathbf{J}}{c^2}\, d\text{Vol} = -\mathbf{P}_{\text{EM}}. \tag{28}$$

A system of currents and charges "at rest" therefore contains zero total momentum, in agreement with one's expectations.

We note that according to eq. (1) the "hidden" mechanical momentum of a quasistatic system can also be written as,

$$\mathbf{P}_{\text{mech,hidden}} = \int \frac{\varrho \mathbf{A}}{c}\, d\text{Vol}, \tag{29}$$

where ϱ is the electric charge density and $\mathbf{A}$ is the Coulomb-gauge vector potential. For further discussion of "hidden" mechanical momentum see, for example, [46].

2.6 Spherical Magnet inside a Spherical Capacitor

A variant of the example of sec. 2.4 is a spherical magnet with a $\sin\theta$ winding of radius b inside a spherical capacitor of radius $a > b$.

The electromagnetic momentum according to the third form of eq. (1) is, recalling eq. (12) for the scalar potential of the capacitor,

$$\begin{aligned}
\mathbf{P}_{\text{EM},3} &= \int_{r=b} \frac{\Phi(b)\mathbf{K}}{c^2}\, d\text{Area} = \frac{3\mathbf{B}_0}{8\pi c} \times \sum_{n \text{ odd}} A_n \left(\frac{b}{a}\right)^n \int_{-1}^{1} 2\pi b^2\, d\cos\theta\; P_n(\cos\theta)\, \hat{\mathbf{r}} \\
&= \frac{3b^2\mathbf{B}_0}{4c} \times \sum_{n \text{ odd}} A_n \left(\frac{b}{a}\right)^n \int_{-1}^{1} d\cos\theta\; P_n(\cos\theta)(\cos\theta\, \hat{\mathbf{z}} + \sin\theta\, \hat{\boldsymbol{\rho}}) \\
&= \frac{b^3\mathbf{B}_0 \times A_1/a\, \hat{\mathbf{z}}}{2c} = \frac{\mathbf{E}_0 \times \mathbf{m}}{c}\,,
\end{aligned} \tag{30}$$

where $\mathbf{m} = b^3\mathbf{B}_0/2$ is the magnetic dipole moment, and $\mathbf{E}_0 = -(A_1/a)\,\hat{\mathbf{z}}$ is the electric field at the origin as given by eq. (14).

To calculate the electromagnetic momentum using the first form of eq. (1), we recall that the vector potential associated with a magnetic moment $\mathbf{m}$ at the origin is,

$$\mathbf{A} = \frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}\,, \tag{31}$$

and that the combined surface charge σ density on the inner and outer surfaces of the capacitor at radius a is given by,

$$\sigma = \frac{E_r(a^+) - E_r(a^-)}{4\pi} = \sum_{n\ \mathrm{odd}} (2n+1)\frac{A_n}{4\pi a}P_n(\cos\theta). \tag{32}$$

Then,

$$\begin{aligned}
\mathbf{P}_{\mathrm{EM},1} &= \int_{r=a} \frac{\sigma\mathbf{A}(a)}{c}\,d\mathrm{Area} = \frac{\mathbf{m}}{4\pi a^2 c}\times\sum_{n\ \mathrm{odd}}(2n+1)\frac{A_n}{a}\int_{-1}^{1} 2\pi a^2\,d\cos\theta\ P_n(\cos\theta)\,\hat{\mathbf{r}} \\
&= \frac{\mathbf{m}}{2c}\times\sum_{n\ \mathrm{odd}}(2n+1)\frac{A_n}{a}\int_{-1}^{1} d\cos\theta\ P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}) \\
&= \frac{\mathbf{m}\times A_1/a\,\hat{\mathbf{z}}}{c} = \frac{\mathbf{E}_0\times\mathbf{m}}{c}\,.
\end{aligned} \tag{33}$$

We partition the calculation of the electromagnetic momentum via form 2 of eq. (1) into the three regions, $r<b$, $b<r<a$ and $r>a$. The electric and magnetic fields for $r<b$ in the present case are the same as those for $r<a$ in sec. 2.4, where we suppose that $\mathbf{B}_0 = B_0\,\hat{\mathbf{y}}$. Hence, from eq. (17) we see that,

$$\mathbf{P}_{\mathrm{EM},2}(r<b) = -\int_{r<b}\frac{E_z B_0}{4\pi c}\,d\mathrm{Vol}\,\hat{\mathbf{x}} = -\frac{E_{z,1}B_0}{4\pi c}\ \mathrm{Vol}\,\hat{\mathbf{x}} = \frac{E_0 b^3 B_0}{3c}\,\hat{\mathbf{x}} = \frac{2\mathbf{E}_0\times\mathbf{m}}{3c}\,. \tag{34}$$

Similarly, the electric and magnetic fields for $r>a$ in the present case are the same as those for $r>b$ in sec. 2.4, so from eq. (25) we see that,

$$\mathbf{P}_{\mathrm{EM},2}(r>a) = \int_{r>a}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} = \frac{a^2 A_1 m}{c}\int_a^\infty \frac{r^2\,dr}{r^6}\,\hat{\mathbf{x}} = \frac{A_1 m}{3ac}\,\hat{\mathbf{x}} = \frac{\mathbf{E}_0\times\mathbf{m}}{3c}\,. \tag{35}$$

For the region $b<r<a$ we again use eqs. (18)-(22), but we note that eq. (23) becomes,

$$\begin{aligned}
\int_0^{2\pi}\mathbf{E}(r<a)\times\mathbf{B}(r>b)\,d\phi &= \frac{\pi m}{r^3}[3(E_r\sin\theta + E_\theta\cos\theta)\sin\theta\cos\theta - E_z(3\sin^2\theta - 2)]\,\hat{\mathbf{x}} \\
&= \frac{\pi m}{r^3}\sum_{n\ \mathrm{odd}} A_n\frac{r^{n-1}}{a^n}\left(-2n\mu P_n + (1-\mu^2)P_n'\right)\,\hat{\mathbf{x}},
\end{aligned} \tag{36}$$

again using eq. (14). The polar part of the volume integral includes the factor,

$$\int_{-1}^{1} d\mu\ \left(-2n\mu P_n + (1-\mu^2)P_n'\right) = \int_{-1}^{1} d\mu\ (-2n\mu P_n + 2\mu P_n) = 0. \tag{37}$$

Thus, the total electromagnetic momentum of a $\sin\theta$ spherical magnet inside a spherical capacitor is the sum of eqs. (34) and (35),

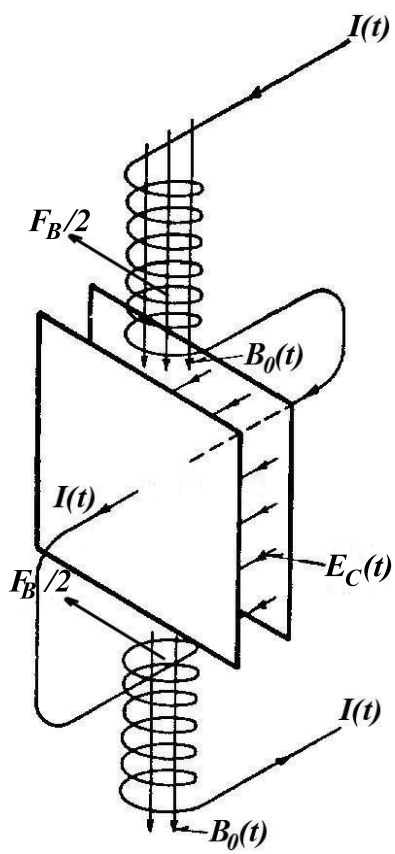
$$\mathbf{P}_{\mathrm{EM},2} = \frac{\mathbf{E}_0 \times \mathbf{m}}{c}\,, \tag{38}$$

in agreement with the calculations (30) and (33) using forms 1 and 3 of eq. (1).

Again, according to the interpretation of momentum being stored in the electromagnetic field, 2/3 of the momentum is within the inner sphere, and 1/3 is outside the outer sphere.

2.7 An Electromagnetic Spaceship?

In the late 1940's Joseph Slepian, a senior engineer at Westinghouse, posed a series of delightful pedagogic puzzles in the popular journal *Electrical Engineering*. One of these concerned how a capacitor in a cylindrical magnetic field might or might not be used to provide a form of rocket propulsion [55].



The current in Slepian's example is sinusoidal at a low enough frequency that radiation is negligible, so that system can be regarded as quasistatic. In this case, the electromagnetic field momentum is always equal and opposite to the "hidden" mechanical momentum, according to a general result of sec. 4.1.4 of [46]. Consequently, the Lorentz force on the system associated with the **E** and **B** field induced by the oscillating **B** and **E** fields are always equal and opposite to the "hidden" momentum forces associated with the oscillatory

"hidden" momentum, and the total momentum of the system remains constant (no rocket propulsion).[12]

A Appendix: Additional Calculations for an Electric Dipole

If we evaluate the electromagnetic momentum for the electric dipole $\mathbf{p} = q\mathbf{d}$ in a long solenoid using either the second or third forms of eq. (1) we obtain divergent contributions of opposite sign from the two charges $\pm q$. It is delicate to obtain the finite resultant of these canceling divergences.

Section A.1 presents a calculation in which a judicious cancelation of divergent terms using the third form of eq. (1) yields the same result as with use of the first form. Section A.2 avoids the problem of divergences in form 3 by consideration of a point dipole at the origin. In sec. A.3 the long solenoid magnet is replaced by a spherical magnet with a $\sin\theta$ winding.

A.1 Calculation Using Form 3 of Eq. (1) for a Long Solenoid

We first consider the third form of eq. (1). The current in the solenoid of radius b whose symmetry axis is the z-axis can be described by the surface current density vector $\mathbf{K} = cB_0\,\boldsymbol{\phi}/4\pi$. Then for a charge q at position $\mathbf{d} = (d, 0, z)$, where $d < b$, only the y-component of $\int \Phi\mathbf{K}\,d\text{Area}$ is nonzero, and we find using Dwight 200.01 [57],[13]

$$\begin{aligned}
\mathbf{P}_{\text{EM},3} &= \hat{\mathbf{y}} \int \frac{\Phi K_y}{c^2}\, d\text{Area} = \hat{\mathbf{y}} \int_0^{2\pi} b\, d\phi \int_{-\infty}^{\infty} dz\, \frac{qK\cos\phi}{c^2\sqrt{b^2 - 2bd\cos\phi + d^2 + z^2}} \\
&= \frac{dqB_0\,\hat{\mathbf{y}}}{4\pi c} \int_0^{2\pi} \cos\phi\, d\phi\, \lim_{z\to\infty} \Big[\ln(z + \sqrt{b^2 - 2bd\cos\phi + d^2 + z^2}) \\
&\qquad\qquad - \ln(-z + \sqrt{b^2 - 2bd\cos\phi + d^2 + z^2})\Big] \\
&= -\frac{dqB_0\,\hat{\mathbf{y}}}{4\pi c} \int_0^{2\pi} \cos\phi\, d\phi\, \Big[\ln(b^2 - 2bd\cos\phi + d^2) \\
&\qquad\qquad -2\lim_{z\to\infty} \ln(z + \sqrt{b^2 - 2bd\cos\phi + d^2 + z^2})\Big]. \qquad (39)
\end{aligned}$$

The divergent term in the last line of eq. (39) is independent of d, so when we add the contribution of charge $-q$ at some other position d', we argue that the divergences cancel. If so, we continue the evaluation of the finite part of eq. (39), integrating by parts and then using Dwight 859.131 [57],

$$\mathbf{P}_{\text{EM},3,\text{finite part}} = -\frac{dqB_0\,\hat{\mathbf{y}}}{4\pi c} \int_0^{2\pi} \cos\phi\, d\phi\, \ln(b^2 - 2bd\cos\phi + d^2)$$

[12] See also [56].

[13] We neglect the effect of the variation of electric potential along the wire of the solenoid winding in case this wire has finite conductivity. We have discussed the electromagnetic momentum of circuits of finite conductivity in [34].

$$= \frac{b^2 q d B_0\,\hat{\mathbf{y}}}{2\pi c}\int_0^{2\pi} d\phi\,\frac{\sin^2\phi}{b^2-2bd\cos\phi+d^2} = \frac{qdB_0\,\hat{\mathbf{y}}}{2c}\,. \tag{40}$$

Then, for an electric dipole $\mathbf{p}$ with charges $\pm q$ at $(d^+, 0, z^+)$ and $(d^-, 0, z^-)$, eq. (40) gives

$$\mathbf{P}_{\mathrm{EM},3} = \frac{q(d^+-d^-)B_0\,\hat{\mathbf{y}}}{2c} = \frac{\mathbf{B}_0\times\mathbf{p}}{2c}\,, \tag{41}$$

in agreement with eq. (3).

A.2 Calculation for a Point Dipole via Form 3 of Eq. (1)

If we use form 3 of eq. (1) to evaluate the electromagnetic momentum of a point dipole in a uniform magnetic field, we again obtain eq. (3), without having to cancel divergent terms. For simplicity, we suppose that the dipole is at the origin, with moment $\mathbf{p} = p\,\hat{\mathbf{x}}$. Then, the electric potential of this dipole at position $\mathbf{r}$ is $\Phi = p\cos\theta/r^2$, where θ is the angle between vector $\mathbf{r}$ and dipole moment $\mathbf{p}$. Point $\mathbf{r} = (b, \phi, z)$ on the solenoid winding of radius a has rectangular coordinates $(b\cos\phi, b\sin\phi, z)$, so that $r = \sqrt{b^2+z^2}$ and $\cos\theta = b\cos\phi/r$. The current in the solenoid is again described by the surface-current-density vector $\mathbf{K} = cB_0\boldsymbol{\phi}/4\pi$, and again only the y-component of $\int \Phi\mathbf{K}\,d\mathrm{Area}$ is nonzero. Hence,

$$\begin{aligned}\mathbf{P}_{\mathrm{EM},3} &= \hat{\mathbf{y}}\int\frac{\Phi K_y}{c^2}\,d\mathrm{Area} = \hat{\mathbf{y}}\int_0^{2\pi} a\,d\phi\int_{-\infty}^{\infty} dz\,\frac{p\cos\theta}{c^2r^2}K\cos\phi \\ &= \frac{b^2B_0p}{4\pi c}\,\hat{\mathbf{y}}\int_0^{2\pi} d\phi\,\cos^2\phi\int_{-\infty}^{\infty}\frac{dz}{(b^2+z^2)^{3/2}} = \frac{B_0p}{2c}\,\hat{\mathbf{y}} = \frac{\mathbf{B}_0\times\mathbf{p}}{2c}\,,\end{aligned} \tag{42}$$

in further agreement with eq. (3).

A.3 Calculation for a Spherical Magnet Using Form 3

The use of a long solenoid as a model of a uniform magnetic field involves unrealistic conditions at "infinity", which may lead to subtle inaccuracies in some calculations of the electromagnetic momentum. A better model for a uniform magnetic field $\mathbf{B}_0$ may be a sphere of (large) radius b on which there exists a surface current density that varies as $\mathbf{K} = 3c\mathbf{B}_0\times\hat{\mathbf{r}}/8\pi$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the sphere. The magnetic field is uniform within the sphere, while outside the sphere the field is that of the magnetic dipole $\mathbf{m} = b^3\mathbf{B}_0/2$ located at the center of the sphere.

We first consider a charge q at position $d/2\,\hat{\mathbf{z}}$ where $d/2 < a$. Then, the electric scalar potential of a charge q at position $d/2\,\hat{\mathbf{z}}$ where $d/2 < b$, can be written in a spherical coordinate system (r, θ, ϕ) for $r > d/2$ as,

$$\Phi_q(r > d/2) = \frac{2q}{d}\sum_n\left(\frac{d}{2r}\right)^{n+1} P_n(\cos\theta), \tag{43}$$

where P_n is a Legendre polynomial.

Then, the electric dipole $\mathbf{p} = qd\,\hat{\mathbf{z}}$ consisting of charges $\pm q$ at positions $\pm d/2\,\hat{\mathbf{z}}$ has potential,

$$\Phi_p(r > d/2) = \frac{4q}{d} \sum_{n \text{ odd}} \left(\frac{d}{2r}\right)^{n+1} P_n(\cos\theta), \tag{44}$$

recalling that $P_n(-\mu) = (-1)^n P_n(\mu)$.

The electromagnetic momentum associated with the electric dipole $\mathbf{p}$ and the spherical magnet is, according to form 3 of eq. (1),

$$\begin{aligned}
\mathbf{P}_{\text{EM},3} &= \int \frac{\Phi_p(b)\mathbf{K}}{c^2}\, d\text{Area} = \frac{4q}{d}\frac{3\mathbf{B}_0}{8\pi c} \times \sum_{n \text{ odd}} \left(\frac{d}{2b}\right)^{n+1} \int_{-1}^{1} 2\pi b^2\, d\cos\theta\; P_n(\cos\theta)\,\hat{\mathbf{r}} \\
&= \frac{3b^2 q\mathbf{B}_0}{cd} \times \sum_{n \text{ odd}} \left(\frac{d}{2b}\right)^{n+1} \int_{-1}^{1} d\cos\theta\; P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}) \\
&= \frac{\mathbf{B}_0 \times qd\,\hat{\mathbf{z}}}{2c} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}\,,
\end{aligned} \tag{45}$$

in agreement with eq. (3).

B Appendix: Transient Analysis

B.1 Thomson's 1904 Analysis of a Magnet plus Electric Charge

In 1904, J.J. Thomson considered a "magnet" (Ampèrian magnetic dipole $\mathbf{m}$) together with an external electric charge [25, 41, 42, 43]. He first deduced the field momentum of the system as,

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{m}}{c}\,, \tag{46}$$

using the second form of eq. (1), which form he had invented in 1891 [23, 24]. He then confirmed eq. (1) using the first (Maxwell) form of eq. (1).

On p. 348 of [42] he remarked, in effect, that his first argument that led to eq. (46) suggested the field momentum is associated with the magnetic dipole, while his second (Maxwell) argument suggested it is associated with the electric charge. He then noted that if the Ampèrian magnetic dipole were a small permanent magnet (in the field of an electric charge), and this magnet were demagnetized by "tapping," the magnet would acquire the initial momentum (46) according to his first argument, while the electric charge should aquire this momentum according to his second (Maxwell) argument.

He did not conclude that these contradictory statements imply the total momentum of the system must be zero (when it is "at rest") [45], such that there exists a "hidden" mechanical momentum [44] in the moving charges of the magnet, equal and opposite to the

field momentum of the system,[14,15,16,17]

$$\mathbf{P}_{\rm mag,hidden} = \frac{\mathbf{m}\times\mathbf{E}}{c} = -\mathbf{P}_{\rm EM}. \tag{47}$$

Then, if the field momentum vanishes the "hidden" mechanical momentum does also, and the total momentum of the system remains zero.

For Thomson's particular example, the magnetic moment drops to zero while the static electric field of the charge is unchanged. The latter field exerts no Lorentz force on the (electrically neutral) magnetic dipole, so there is no change in its total mechanical momentum,

$$\mathbf{F}_{\rm mag,Lorentz} = 0 = \frac{d\mathbf{P}_{\rm mag}}{dt} = \frac{d}{dt}(\mathbf{P}_{\rm mag,overt} + \mathbf{P}_{\rm mag,hidden}) = M_{\rm mag}\frac{d\mathbf{v}_{\rm mag}}{dt} + \frac{d\mathbf{P}_{\rm mag,hidden}}{dt}\,, \tag{48}$$

where the "overt" momentum of the magnet is its mass $M_{\rm mag}$ times the velocity $\mathbf{v}_{\rm mag}$ of its center of mass. The "overt" mechanical momentum of the magnet/dipole changes according to,[18]

$$M_{\rm mag}\frac{d\mathbf{v}_{\rm mag}}{dt} = \frac{d\mathbf{P}_{\rm mag,overt}}{dt} = \mathbf{F}_{\rm mag,Lorentz} - \frac{d\mathbf{P}_{\rm mag,hidden}}{dt} = -\frac{d\mathbf{P}_{\rm mag,hidden}}{dt}\,, \tag{49}$$

$$M_{\rm mag}\mathbf{v}_{\rm mag,final} = \mathbf{P}_{\rm mag,overt,final} = \mathbf{P}_{\rm mag,hidden,initial} = \frac{\mathbf{m}\times\mathbf{E}}{c}\,. \tag{50}$$

Meanwhile, the decreasing magnetic moment leads to an induced electric field at the charge q, such that the force on the charge (which has no magnetic moment and no "hidden" momentum) is,

$$\mathbf{F} = q\mathbf{E}_{\rm ind} = -\frac{q}{c}\frac{\partial\mathbf{A}}{\partial t} = -\frac{q\,\dot{\mathbf{m}}\times\mathbf{r}}{cr^3} = \frac{\dot{\mathbf{m}}\times\mathbf{E}}{c} = -\frac{d\mathbf{P}_{\rm mag,overt}}{dt}\,. \tag{51}$$

The final ("overt") momentum of the charge is,

$$M_q\mathbf{v}_{q,\rm final} = \mathbf{P}_{q,\rm overt,final} = -\frac{\mathbf{m}\times\mathbf{E}}{c} = -\mathbf{P}_{\rm mag,overt,final} = \mathbf{P}_{\rm EM,initial}, \tag{52}$$

The final total momentum of the system is zero, while the magnet and the charge have equal and opposite final momenta.

[14]The form (47) was first given in eq. (7.85) of [47]. See also Ex. 12.13 of [48] and [49].

[15]A loop of area A that carries current I has magnetic moment $m = IA/c$, so the "hidden" momentum (47) of the Ampèrian magnetic dipole (and the field momentum (46), which is the negative of eq. (47)) is an effect of order $1/c^2$ and can be called "relativistic." Of course, the latter notion was not yet well established 1904.

[16]The stable electric charges found in Nature, electrons and protons, also have magnetic moments, and can possess "hidden" momentum. See, for example, [50, 51].

In this note we suppose that electric charges do not have magnetic moments.

[17]For discussion of a general definition of "hidden" momentum, see [46].

[18]When Shockley coined the term "hidden momentum" [44], he was also concerned with "hidden-momentum forces" that arise when changes occur in examples like that of the present note. The term $-d\mathbf{P}_{\rm mech,hidden}/dt$ was called the "hidden-momentum force" by Shockley [44], but is only occasionally mentioned in the literature. See, for example, sec. IV of [52], p. 53 of [53], and sec. 2.5 of [54]. *Some care is required in reading [52], where the distinction between Ampèrian and Gilbertian magnetic dipoles was not always observed.*

B.2 Transient Analysis of a Magnet and Capacitor

October 2015. Appendix B.2 was inspired by comments from B.Y.-K. Hu [39].

In Appendix B.2, we consider a system consisting of an electrically neutral "magnet" that has initial (Ampèrian) magnetic dipole moment $\mathbf{m}_i$ and a "capacitor" with initial (Gilbertian) electric dipole moment $\mathbf{p}_i$. These moments are both nonzero initially, and both the magnet and capacitor are initially at rest. The initial total momentum of the system is zero,

$$\mathbf{P}_{\text{total,initial}} = 0 = \mathbf{P}_{\text{mech,initial}} + \mathbf{P}_{\text{EM,initial}} = \mathbf{P}_{\text{mech,hidden,initial}} + \mathbf{P}_{\text{EM,initial}}, \tag{53}$$

and we suppose that the initial field momentum is well approximated by the expression [25, 41, 42, 43],

$$\mathbf{P}_{\text{EM,initial}} = \frac{\mathbf{E}_i \times \mathbf{m}_i}{c} \ \left(= -\mathbf{P}_{\text{mech,hidden,initial}}\right), \tag{54}$$

where $\mathbf{E}_i$ is the initial electric field at the magnet due to the capacitor. For this to be so, the capacitor should be outside the magnet, and their separation should be large compared to their characteristic lengths. Then, according to eq. (46), the electromagnetic field momentum of the system is $\mathbf{P}_{\text{EM}} = [3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}] \times \mathbf{m}/cr^3$ ($\neq \mathbf{B}\times\mathbf{p}/2c = -[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}] \times \mathbf{p}/2cr^3$), where $\mathbf{r}$ is distance vector from $\mathbf{p}$ to $\mathbf{m}$.

The field momentum of the electric dipole $\mathbf{p} = q(\mathbf{d}^+ - \mathbf{d}^-)$ in a magnetic field $\mathbf{B}$ can also be computed from the vector potential $\mathbf{A}(\mathbf{r}) = \int [\mathbf{J}(\mathbf{r}')/c\,|\mathbf{r}-\mathbf{r}'|]\,d\text{Vol}'$ (in the Coulomb gauge), due to the current density $\mathbf{J}$ in the magnet that produces $\mathbf{B}$, according to Maxwell's form (1),

$$\begin{aligned}\mathbf{P}_{\text{EM}} &= \int \frac{\varrho\mathbf{A}}{c}\,d\text{Vol} = q\frac{\mathbf{A}(\mathbf{d}^+) - \mathbf{A}(\mathbf{d}^-)}{c} = (\mathbf{p}\cdot\boldsymbol{\nabla})\frac{\mathbf{A}}{c} = \frac{\mathbf{B}\times\mathbf{p}}{c} + \boldsymbol{\nabla}\left(\frac{\mathbf{p}\cdot\mathbf{A}}{c}\right) \\ &= \frac{\mathbf{B}\times\mathbf{p}}{c} + \frac{\mathbf{p}}{c^2}\cdot\int \mathbf{J}(\mathbf{r}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d\text{Vol}' = \frac{\mathbf{B}\times\mathbf{p}}{c} - \frac{\mathbf{p}}{c^2}\cdot\int \mathbf{J}(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,d\text{Vol}',\end{aligned} \tag{55}$$

where the fourth form was deduced as eq. (65) of [38], and the fifth form follows from the vector-calculus identity for $\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{A})$, noting that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and that the operator $\boldsymbol{\nabla}$ does not act on the dipole moment $\mathbf{p} = q(\mathbf{d}^+ - \mathbf{d}^-)$. The last form is due to B.Y.-K. Hu, eq. (9d) of [39]. In eq. (55), $\mathbf{r}$ is the position of the capacitor/electric dipole moment $\mathbf{p}$.

We consider two transient scenarios: the magnet current drops to zero while the capacitor remains charged; and the capacitor discharges while the magnet current remains constant.

B.2.1 The Magnet Current (and Dipole Moment m) Drop to Zero

This case is similar to that of sec. B.1, where the magnet current also dropped to zero.

When the current in the magnet is changing, there exists a transient electric field given by,

$$\mathbf{E}_{\text{transient}} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\,, \tag{56}$$

where $\mathbf{A}$ is the vector potential (in the Coulomb gauge) associated with the magnet current. This transient electric field exerts a force with density $\varrho\mathbf{E}_{\text{transient}}$ on the charge density ϱ of

the capacitor, so the capacitor takes on momentum,

$$\mathbf{P}_{\text{cap,final}} = M_{\text{cap}}\mathbf{v}_{\text{cap,final}} = \int \mathbf{F}_{\text{cap}}\, dt = -\frac{1}{c}\int \varrho\, d\text{Vol} \int \frac{\partial \mathbf{A}}{\partial t}\, dt = \int \frac{\varrho \mathbf{A}_i}{c}\, d\text{Vol} = \mathbf{P}_{\text{EM,initial}}. \quad (57)$$

That is, the initial field momentum becomes converted to mechanical momentum of the capacitor as the current and the magnetic moment $\mathbf{m}$ drop to zero.[19]

The electric field of the capacitor exerts no force on the electrically neutral magnet, $\mathbf{F}_{\text{mag}} = 0$, so it might seem that the magnet ends up with zero momentum, once the initial "hidden" mechanical momentum, $\mathbf{P}_{\text{mech,hidden,initial}} = \mathbf{E} \times \mathbf{m}/c$, has fallen to zero. If so, the total final momentum of the system would be nonzero, and momentum would not be conserved.

However, we should consider Newton's law for the magnet to read,

$$\mathbf{F}_{\text{mag}} = \frac{d\mathbf{P}_{\text{mag}}}{dt} = \frac{d}{dt}(M_{\text{mag}}\mathbf{v}_{\text{mag}} + \mathbf{P}_{\text{mech,hidden}}), \quad (58)$$

where M_{mag} and $\mathbf{v}_{\text{mag}}$ are the mass and velocity of the magnet, in that the "hidden" mechanical momentum of the system is associated with the moving charges of the magnet current. Hence, the motion of the magnet is related by,

$$M_{\text{mag}}\frac{d\mathbf{v}_{\text{mag}}}{dt} = \mathbf{F}_{\text{mag}} - \frac{d}{dt}\mathbf{P}_{\text{mech,hidden}} \quad (59)$$

So, even though $\mathbf{F}_{\text{mag}} = 0$, the magnet ends up with a small final momentum,[20]

$$\mathbf{P}_{\text{mag,final}} = M_{\text{mag}}\mathbf{v}_{\text{mag,final}} = \mathbf{P}_{\text{mech,hidden,initial}} = \frac{\mathbf{E}_i \times \mathbf{m}_i}{c} = -\mathbf{P}_{\text{EM,initial}} = -\mathbf{P}_{\text{cap,final}}, \quad (60)$$

such that the final total momentum is zero, as expected from momentum conservation,

$$\mathbf{P}_{\text{total,final}} = \mathbf{P}_{\text{mag,final}} + \mathbf{P}_{\text{cap,final}} = \frac{\mathbf{E}_i \times \mathbf{m}_i}{c} + \frac{\mathbf{m}_i \times \mathbf{E}_i}{c} = 0. \quad (61)$$

B.2.2 The Capacitor Discharges (and the Dipole Moment p Drops to Zero)

If instead the capacitor discharges while the magnet current is held constant (by a "battery"), there exists a transient current density associated with the capacitor, which can be represented as,

$$\mathbf{J}_{\text{cap}} = \frac{d\mathbf{p}}{dt}\delta^3(\mathbf{r} - \mathbf{r}_{\text{cap}}), \quad (62)$$

for a small (pointlike) capacitor. The magnetic field $\mathbf{B}_i$ of the magnet at the location of the capacitor exerts a Lorentz force on this transient current density,

$$\mathbf{F}_{\text{cap}} = \int \frac{\mathbf{J}_{\text{cap}}}{c} \times \mathbf{B}_i\, d\text{Vol} = \frac{d\mathbf{p}}{dt} \times \frac{\mathbf{B}_i}{c}, \quad (63)$$

[19] The moving capacitor has a small magnetic moment, $\mathbf{m}_{\text{cap}} \approx \mathbf{p}_{\text{cap}} \times \mathbf{v}_{\text{cap}}/c$ [58], whose tiny effect is ignored here.

[20] An interpretation of eqs. (59)-(60) is that as the "hidden" mechanical momentum drops to zero (according to eq. (28), with zero final current in the magnet), it becomes converted (by "hidden-momentum forces" [44] not described by $\mathbf{F}_{\text{mag}}$) to "ordinary" (or "overt") mechanical momentum associated with motion of the magnet's center of mass.

such that the final momentum of the capacitor (whose initial moment is $\mathbf{p}_i$ and whose final moment is zero) is,

$$\mathbf{P}_{\text{cap,final}} = \int \mathbf{F}_{\text{cap}}\, dt = -\frac{\mathbf{p}_i \times \mathbf{B}_i}{c} = \frac{\mathbf{B}_i \times \mathbf{p}_i}{c} = M_{\text{cap}}\mathbf{v}_{\text{cap,final}}\,. \tag{64}$$

In addition, the current density (62) of the discharging capacitor generates a transient magnetic field according to the Biot-Savart law (ignoring radiation and retardation),

$$\mathbf{B}_{\text{cap}}(\mathbf{r}) = \int \frac{\mathbf{J}_{\text{cap}}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{c\,|\mathbf{r}-\mathbf{r}'|^3}\, d\text{Vol}'_{\text{cap}} = \frac{d\mathbf{p}}{dt} \times \frac{\mathbf{r}-\mathbf{r}_{\text{cap}}}{c\,|\mathbf{r}-\mathbf{r}_{\text{cap}}|^3}\,. \tag{65}$$

This transient field exerts a Lorentz force on the (constant) current density $\mathbf{J}_i$ of the magnet,

$$\mathbf{F}_{\text{mag}} = \int \frac{\mathbf{J}_i \times \mathbf{B}_{\text{cap}}}{c}\, d\text{Vol}_{\text{mag}} = \frac{d\mathbf{P}_{\text{mag}}}{dt} = \frac{d}{dt}(M_{\text{mag}}\mathbf{v}_{\text{mag}} + \mathbf{P}_{\text{mag,hidden}}), \tag{66}$$

recalling eq. (58). Hence (ignoring the small change in position of the magnet as the capacitor discharges),

$$\begin{aligned}
\int \mathbf{F}_{\text{mag}}\, dt &= \int \mathbf{J}_i \times \left(\mathbf{p}_i \times \frac{\mathbf{r}-\mathbf{r}_{\text{cap}}}{c^2\,|\mathbf{r}-\mathbf{r}_{\text{cap}}|^3}\right) d\text{Vol}_{\text{mag}} \\
&= \mathbf{p}_i \int \mathbf{J}_i \cdot \frac{\mathbf{r}-\mathbf{r}_{\text{cap}}}{c^2\,|\mathbf{r}-\mathbf{r}_{\text{cap}}|^3}\, d\text{Vol}_{\text{mag}} - \int \mathbf{J}_i \cdot \mathbf{p}_i \frac{\mathbf{r}-\mathbf{r}_{\text{cap}}}{c^2\,|\mathbf{r}-\mathbf{r}_{\text{cap}}|^3}\, d\text{Vol}_{\text{mag}} \\
&= -\int \mathbf{J}_i \cdot \mathbf{p}_i \frac{\mathbf{r}_{\text{cap}}-\mathbf{r}}{c^2\,|\mathbf{r}_{\text{cap}}-\mathbf{r}|^3}\, d\text{Vol}_{\text{mag}} \\
&= M_{\text{mag,final}}\mathbf{v}_{\text{mag,final}} + \mathbf{P}_{\text{mag,hidden,final}} - M_{\text{mag,initial}}\mathbf{v}_{\text{mag,initial}} - \mathbf{P}_{\text{mag,hidden,initial}} \\
&= M_{\text{mag,final}}\mathbf{v}_{\text{mag,final}} - \mathbf{P}_{\text{mech,hidden,initial}} = \mathbf{P}_{\text{mag,final}} - \mathbf{P}_{\text{mag,hidden,initial}},
\end{aligned} \tag{67}$$

noting that the final "hidden" momentum is zero according to eq. (29) since $\varrho_{\text{final}} = 0$, and,

$$\begin{aligned}
\int \mathbf{J} \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d\text{Vol}_{\text{mag}} &= -\int \mathbf{J} \cdot \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d\text{Vol}_{\text{mag}} \\
= \int \frac{\boldsymbol{\nabla}\cdot\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\, d\text{Vol}_{\text{mag}} - \int \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right) d\text{Vol}_{\text{mag}} &= -\oint \frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \cdot d\mathbf{Area} \to 0,
\end{aligned} \tag{68}$$

for a bounded, static current density, which obeys $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$.[21]

We also recall that the initial "hidden" momentum is equal and opposite to the initial field momentum, so that according to eq. (55),

$$\mathbf{P}_{\text{EM,initial}} = \mathbf{B}_i \times \mathbf{p}_i - \int \mathbf{J}_i \cdot \mathbf{p}_i \frac{\mathbf{r}_{\text{cap}}-\mathbf{r}'}{c^2\,|\mathbf{r}_{\text{cap}}-\mathbf{r}'|^3}\, d\text{Vol}'_{\text{mag}} = -\mathbf{P}_{\text{mag,hidden,initial}}. \tag{69}$$

[21] For an "infinite" solenoid, with $\mathbf{J}$ perpendicular to its axis, $\mathbf{J}\cdot d\mathbf{Area} = 0$ for spherical surfaces centered on a point on the solenoid axis, so eq. (68) holds for this case also.

Altogether, we have,

$$\begin{aligned}\mathbf{P}_{\text{mag,final}} &= M_{\text{mag}}\mathbf{v}_{\text{mag,final}} = -\int \mathbf{J}_i \cdot \mathbf{p}_i \frac{\mathbf{r}_{\text{cap}} - \mathbf{r}'}{c^2 \left|\mathbf{r}_{\text{cap}} - \mathbf{r}'\right|^3} \, d\text{Vol}'_{\text{mag}} + \mathbf{P}_{\text{mech,hidden,initial}} \\ &= -\int \mathbf{J}_i \cdot \mathbf{p}_i \frac{\mathbf{r}_{\text{cap}} - \mathbf{r}'}{c^2 \left|\mathbf{r}_{\text{cap}} - \mathbf{r}'\right|^3} \, d\text{Vol}'_{\text{mag}} - \mathbf{P}_{\text{EM,initial}} = -\mathbf{B}_i \times \mathbf{p}_i, \end{aligned} \tag{70}$$

and the total, final momentum is,

$$\mathbf{P}_{\text{total,final}} = \mathbf{P}_{\text{mag,final}} + \mathbf{P}_{\text{cap,final}} = -\mathbf{B}_i \times \mathbf{p}_i + \mathbf{B}_i \times \mathbf{p}_i = 0, \tag{71}$$

as expected.

Acknowledgment

The author wishes to thank Vladimir Hnizdo and Dave Jackson for discussions on electromagnetic momentum and "hidden" mechanical momentum.

References

[1] H. von Helmholtz, *über discontinuirliche Flüssigkeits-Bewegungen*, Monatsberichte der Königl. Akad. Wissen. (Berlin) (April 23, 1868), p. 215,
http://kirkmcd.princeton.edu/examples/fluids/helmholtz_mkrawb_215_68.pdf
See also Art. 202, vol. I of J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed. (Clarendon Press, 1892, 1904), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf

[2] G. Kirchhoff, *Zur Theorie des Condensators*, Monatsberichte der Königl. Akad. Wissen. (Berlin), pp. 144-162 (1877), http://kirkmcd.princeton.edu/examples/EM/kirchhoff_mbkpawb_144_78.pdf

[3] J.W. Nicholson, *The Electrification of Two Parallel Circular Discs*, Phil. Trans. Roy. Soc. London **A224**, 303 (1924),
http://kirkmcd.princeton.edu/examples/EM/nicholson_ptrsla_224_303_24.pdf

[4] E.M. Love, *The Electrostatic Field of Two Equal Circular Co-axial Conducting Disks*, Quart. J. Appl. Math. **2**, 428 (1949),
http://kirkmcd.princeton.edu/examples/EM/love_qjam_2_428_49.pdf

[5] W.J. Atkinson, J.H. Young and I.A. Brezovich, *An analytic solution for the potential due to a circular parallel plate capacitor*, J. Phys. A **16**, 2837 (1983),
http://kirkmcd.princeton.edu/examples/EM/atkinson_jpa_16_2837_83.pdf

[6] B.D. Hughes, *On the potential due to a circular parallel plate capacitor*, J. Phys. A **17**, 1385 (1984), http://kirkmcd.princeton.edu/examples/EM/hughes_jpa_17_1385_84.pdf

[7] D.F. Bartlett and T.R. Cole, *The circular parallel plate capacitor: a numerical solution for the potential*, J. Phys. A **18**, 1337 (1985),
http://kirkmcd.princeton.edu/examples/EM/bartlett_jpa_18_1337_85.pdf

[8] H.J. Wintle and S. Kurylowicz, *Edge corrections for strip and disc capacitors*, IEEE Trans. Instrum. Meas. **IM-34**, 41 (1985), http://kirkmcd.princeton.edu/examples/EM/wintle_ieeetim_34_41_85.pdf

[9] G.J. Sloggettt, N.G. Barton and S.J. Spencer, *Fringing fields in disc capacitors*, J. Phys. A **19**, 2725 (1986), http://kirkmcd.princeton.edu/examples/EM/sloggett_jpa_19_2725_86.pdf

[10] H. Nishiyama and M. Nakamura, *Capacitance of Disk Capacitors*, IEEE Trans. Components, Packaging and Manufacturing Tech. **16**, 360 (1993), http://kirkmcd.princeton.edu/examples/EM/nishiyama_ieeetcpmt_16_360_93.pdf

[11] H. Nishiyama and M. Nakamura, *Form and Capacitance of Parallel-Plate Capacitors*, IEEE Trans. Components, Packaging and Manufacturing Tech. A **17**, 477 (1994), http://kirkmcd.princeton.edu/examples/EM/nishiyama_ieeetcpmta_17_477_94.pdf

[12] G.T. Carlson and B.L. Illman, *The circular disk parallel plate capacitor*, Am. J. Phys. **62**, 10991105 (1994), http://kirkmcd.princeton.edu/examples/EM/carlson_ajp_62_1099_94.pdf

[13] G.W Parker, *Electric field outside a parallel plate capacitor*, Am. J. Phys. **70**, 502 (2002), http://kirkmcd.princeton.edu/examples/EM/parker_ajp_70_502_02.pdf

[14] F.B. Johnson, B.L. Cragin and R.R. Hodges, *Electromagnetic momentum density and the Poynting vector in static fields*, Am. J. Phys. **62**, 33 (1994), http://kirkmcd.princeton.edu/examples/EM/johnson_ajp_62_33_94.pdf

[15] J.D. Jackson, *Relation between Interaction terms in Electromagnetic Momentum* $\int d^3\mathbf{x}\,\mathbf{E}\times\mathbf{B}/4\pi c$ *and Maxwell's* $e\mathbf{A}(\mathbf{x},t)/c$, *and Interaction terms of the Field Lagrangian* $\mathcal{L}_{\rm em}=\int d^3\mathbf{x}\,[E^2-B^2]/8\pi$ *and the Particle Interaction Lagrangian,* $\mathcal{L}_{\rm int}=e\phi-e\mathbf{v}\cdot\mathbf{A}/c$ (May 8, 2006), http://kirkmcd.princeton.edu/examples/EM/jackson_050806.pdf

[16] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf

[17] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[18] M. Faraday, *Experimental Researches in Electricity.—Thirteenth Series*, Phil. Trans. Roy. Soc. London **128**, 125 (1838), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_128_125_38.pdf

[19] M. Faraday, *Experimental Researches in Electricity.—Fourteenth Series*, Phil. Trans. Roy. Soc. London **128**, 265 (1838), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_128_265_38.pdf

[20] M. Faraday, *On the Physical Character of the Lines of Magnetic Force*, Phil. Mag. **3**, 401 (1852), kirkmcd.princeton.edu/examples/EM/faraday_pm_3_401_52.pdf

[21] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[22] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[23] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[24] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893),
http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf

[25] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[26] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[27] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[28] Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052 (1988), http://kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf

[29] R.H. Romer, *Electromagnetic field momentum*, Am. J. Phys. **63**, 777 (1995),
http://kirkmcd.princeton.edu/examples/EM/romer_ajp_63_777_95.pdf

[30] K.T. McDonald, *The Transverse Momentum of an Electron in a Wave* (Nov. 15, 1998),
http://kirkmcd.princeton.edu/examples/transmom2.pdf

[31] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002),
http://kirkmcd.princeton.edu/examples/hidden.pdf

[32] K.T. McDonald, *Onoochin's Paradox* (Jan. 1, 2006),
http://kirkmcd.princeton.edu/examples/onoochin.pdf

[33] K.T. McDonald, *McKenna's Paradox: Charged Particle Exiting the Side of a Solenoid Magnet* (April 12, 2006), http://kirkmcd.princeton.edu/examples/mckenna.pdf

[34] K.T. McDonald, *Momentum in a DC Circuit* (May 26, 2006),
http://kirkmcd.princeton.edu/examples/loop.pdf

[35] K.T. McDonald, *Cullwick's Paradox: Charged Particle on the Axis of a Toroidal Magnet* (June 4, 2006), http://kirkmcd.princeton.edu/examples/cullwick.pdf

[36] K.T. McDonald, *Slepian's Electromagnetic Spaceship* (June 18, 2006),
http://kirkmcd.princeton.edu/examples/slepian.pdf

[37] K.T. McDonald, *Ph501 Electrodynamics, Problem Set 4* (1999), http://kirkmcd.princeton.edu/examples/ph501set4.pdf

[38] D.J. Griffiths, *Dipoles at rest*, Am. J. Phys. **60**, 979 (1992), http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_60_979_92.pdf

[39] B.Y.-K. Hu, *On electromagnetic momentum of an electric dipole in a magnetic field* (Sept. 19, 2015), http://arxiv.org/abs/1408.4144

[40] A. Gsponer, *On the electromagnetic momentum of static charge and steady current distributions*, Eur. J. Phys. **28**, 1021 (2007), http://kirkmcd.princeton.edu/examples/EM/gsponer_ejp_28_1021_07.pdf

[41] J.J. Thomson, *Electricity and Matter* (Charles Scribner's Sons, 1904), pp. 25-35, http://kirkmcd.princeton.edu/examples/EM/thomson_electricity_matter_04.pdf

[42] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf

[43] J.J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, 3rd ed. (Cambridge U. Press, 1904), http://kirkmcd.princeton.edu/examples/EM/thomson_EM_3rd_ed_04.pdf

[44] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[45] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[46] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[47] P. Penfield, Jr and H.A. Haus, *Electrodynamics of Moving Media*, p. 215 (MIT, 1967), http://kirkmcd.princeton.edu/examples/EM/penfield_haus_chap7.pdf

[48] D.J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Addison-Wesley, 2012), http://kirkmcd.princeton.edu/examples/EM/griffiths_8.3_12.12.pdf

[49] K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012), http://kirkmcd.princeton.edu/examples/penfield.pdf

[50] K.T. McDonald, *"Hidden" Momentum in a e^+e^- Pair* (Apr. 30, 2015), http://kirkmcd.princeton.edu/examples/e+e-.pdf

[51] J.S. Oliveira Filho and P.L. Saldanha, *Hidden momentum in a hydrogen atom and the Lorentz force law* (Aug. 2, 2015), http://arxiv.org/abs/1507.08953

[52] L. Vaidman, *Torque and force on a magnetic dipole*, Am. J. Phys. **58**, 978 (1990), http://kirkmcd.princeton.edu/examples/EM/vaidman_ajp_58_978_90.pdf

[53] M.G. Calkin, *Lagrangian and Hamiltonian Mechanics* (World Scientific, 1996), http://kirkmcd.princeton.edu/examples/EM/calkin_96_p49.pdf

[54] K.T. McDonald, *Forces on Magnetic Dipoles* (Oct. 26, 2014), http://kirkmcd.princeton.edu/examples/neutron.pdf

[55] J. Slepian, *Electromagnetic Space-Ship*, Electrical Engineering **68**, 145 (1949), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_68_145_49

[56] K.T. McDonald, *No Bootstrap Spaceships* (Aug. 9, 2018), http://kirkmcd.princeton.edu/examples/bootstrap.pdf

[57] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[58] V. Hnizdo and K.T. McDonald, *Fields and Moments of a Moving Electric Dipole* (Nov. 29, 2011), http://kirkmcd.princeton.edu/examples/movingdipole.pdf