Čerenkov Radiation by a Neutron and by an Electric Dipole

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1 Problem

A neutron has no electric charge, but it does have a magnetic moment $\mathbf{m}$. Hence, we can expect an accelerated neutron to emit radiation. Here, we ask whether a neutron will emit Čerenkov radiation when traveling inside a dielectric medium with uniform velocity $v > c/n$, where c is the speed of light in vacuum and n is the index of refraction of the medium?

Also discuss the case of an (electrically neutral) electric dipole, such as a water molecule.

It suffices to suppose that the magnetic moment is parallel to the velocity, that the electric-dipole moment is perpendicular to the velocity, and that both of these are "point" moments.

2 Solution

Towards answering this, we consider the spectrum of energy *vs.* angular frequency ω and solid angle Ω of a pulse of radiation due to electric charge e with time-dependent velocity $\mathbf{v}$,[1]

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\omega^2 n}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\beta})\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2 = \frac{\omega^2 n}{4\pi^2 c}\left|\int_{-\infty}^{\infty} e\boldsymbol{\beta} \times \hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2, \qquad (1)$$

in Gaussian units, where $\mathbf{k}$ is the wave vector with $k = n\omega/c$ in case of a medium with index of refraction n, and hence $\hat{\mathbf{k}} = \hat{\mathbf{n}}$ is the unit vector pointing to the observer. Also, $\boldsymbol{\beta} = \mathbf{v}/c$.

2.1 Čerenkov Radiation by a Neutron

A magnetic moment may be thought of as an electric-current loop, so we need a version of eq. (1) for a current rather than an electric charge, For this we note that for a moving charge,[2]

[1] This approach follows pp. 261-265 of [1]. See also sec. 20-7 of [2], and p. 250 of the author's E&M Lecture 21 [3].

A derivation of eq. (1) in vacuum via the Liènard-Wiechert fields is given in sec. 14.5 of [4].

[2] For a point charge e at the origin is its rest frame we can write its charge density as $\rho^\star = e\,\delta(x^\star)\,\delta(y^\star)\,\delta(z^\star)$, and, of course, the current density is zero, $\mathbf{J}^\star = 0$. In a frame where the charge has velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, the charge and current densities follow from the Lorentz transformations, $\rho = \gamma\rho^\star = \gamma e\,\delta(x^\star)\,\delta(y^\star)\,\delta(z^\star) = \gamma e\,\delta(x)\,\delta(y)\,\delta(\gamma(z - vt)) = e\,\delta(x)\,\delta(y)\,\delta(z - vt)$, and $\mathbf{J} = \gamma\rho^\star v\,\hat{\mathbf{z}} = \gamma e\,\delta(x)\,\delta(y)\,\delta(\gamma(z - vt))v\,\hat{\mathbf{z}} = e\,\delta(x)\,\delta(y)\,\delta(z - vt)v\,\hat{\mathbf{z}} = \rho v\,\hat{\mathbf{z}}$, noting that $\delta(\gamma z) = \delta(z)/\gamma$ since $\int f(z)\,\delta(\gamma z)\,dz = \int (f(z)/\gamma)\,\delta(\gamma z)\,d(\gamma z) = f(0)/\gamma$. That is, while an extended charge density is enhanced by the Lorentz contraction in a frame where the density is in motion, a point charge is not subject to the Lorentz contraction.

$$e\,\boldsymbol{\beta} \to \frac{\mathbf{J}}{c}\,d\mathrm{Vol}, \tag{2}$$

and hence,[3]

$$\frac{dU_\omega}{d\Omega} = \frac{\omega^2 n}{4\pi^2 c^3}\left|\int\int \mathbf{J}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2. \tag{3}$$

For a neutron moving with its magnetic moment $\mathbf{m}$ parallel to its uniform velocity $v\,\hat{\mathbf{z}}$ in a medium of index of refraction n, we use eq. (3), taking the current density to be,[4]

$$\mathbf{J} = c\boldsymbol{\nabla}\times\mathbf{m} = c\boldsymbol{\nabla}\times m_0\,\hat{\mathbf{z}}\,\delta(x)\,\delta(y)\,\delta(z-vt). \tag{4}$$

Hence, from eq. (3),

$$\begin{aligned}
\frac{dU_\omega}{d\Omega} &= \frac{\omega^2 n}{4\pi^2 c}\left|\int\int (\boldsymbol{\nabla}\times\mathbf{m})\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 n}{4\pi^2 c}\left|-\int\int (-i\mathbf{k}\times\mathbf{m})\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n}{4\pi^2 c}\left|\int\int (\hat{\mathbf{k}}\times\mathbf{m})\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n}{4\pi^2 c}\left|\int\int m_0\sin\theta\,\delta(x)\,\delta(y)\,\delta(z-vt)\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n m_0^2\sin^2\theta}{4\pi^2 c}\left|\int e^{i\omega t(1-(nv/c)\cos\theta)}\,dt\right|^2,
\end{aligned} \tag{5}$$

where the second line follows from the first via integration by parts with respect to volume, while in the fifth line we note the $k = n\omega/c$ for waves in a medium of index n, and we take θ as the angle between $\hat{\mathbf{n}}$ and the z-axis.

The remaining integral is the same as in an intermediate step of the computation for Čerenkov radiation by electric charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, as on the top of p. 251 of [3], where the prefactor is $e^2\omega^2 n v^2\sin^2\theta/4\pi^2 c^3$. Hence,

$$\frac{dU_\omega/d\Omega|_{\mathrm{moving\ neutron}}}{dU_\omega/d\Omega|_{\mathrm{moving\ charge}\ e}} = \frac{c^2}{v^2}\frac{k^2 m_0^2}{e^2} = \frac{c^2}{v^2}\frac{m_0^2}{e^2\lambda\!\!\!^{-2}}. \tag{6}$$

Since $m_{\mathrm{neutron}} \approx e\hbar/Mc = e\lambda\!\!\!^{-}_{\mathrm{neutron}}$, where $\lambda\!\!\!^{-}_{\mathrm{neutron}}$ is the Compton wavelength of the neutron, the ratio is approximately $\lambda\!\!\!^{-2}_{\mathrm{neutron}}/\lambda\!\!\!^{-2}$ for the Čerenkov radiation at reduced wavelength $\lambda\!\!\!^{-} = \lambda/2\pi = 1/k$. That is, Čerenkov radiation by a neutron is an extremely weak effect.[5]

[3]Equation (3) also follows from p. 182 of the author's E&M Lecture 15 [5], since $|\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{J})| = |\mathbf{J}\times\hat{\mathbf{k}}|$.

[4]If the magnetic moment has a component perpendicular to its velocity, then it appears to have an electric-dipole moment as well (as reviewed in the Appendix), which also contributes to Čerenkov radiation. This contribution is different for a magnetic moment due to electric currents and one due to opposite magnetic charges (monopoles).

The earliest computations of Čerenkov radiation by neutrons (Frank, 1942) assumed the latter, while it is now believed that the former assumption is more appropriate. See, for example, [6].

[5]For another exotic Čerenkov effect, involving "light bullets", see [7].

The author has made an experimental demonstration of the interference between Čerenkov radiation and synchrotron radiation by a relativistic electron moving on a circular path in a gas [8].

2.2 Čerenkov Radiation by a Point Electric Dipole

We now consider an electrically neutral particle with electric-dipole moment $\mathbf{p}_0$ in its rest frame, and zero magnetic moment there. For simplicity, we also suppose $\mathbf{p}_0$ to be perpendicular to the lab-frame velocity $\mathbf{v}$, where $v > c/n$ in the medium of index of refraction n. Then, from eq. (19) of the Appendix, the lab-frame current density is,

$$\mathbf{J} = -\mathbf{v}\,(\mathbf{p}_0 \cdot \boldsymbol{\nabla})\,\delta(x)\,\delta(y)\,\delta(z - vt) = \rho\mathbf{v}, \tag{7}$$

and the frequency-angle spectrum of the radiation is given by,

$$\begin{aligned}
\frac{dU_\omega}{d\Omega} &= \frac{\omega^2 n}{4\pi c^3}\left|\int\int \mathbf{J}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 n}{4\pi^2 c^3}\left|\int\int \mathbf{v}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}(\mathbf{p}_0\cdot\boldsymbol{\nabla})\,\delta(x)\,\delta(y)\,\delta(z-vt)\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 n}{4\pi^2 c^3}\left|-\int\int \mathbf{v}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,(\mathbf{p}_0\cdot -i\mathbf{k})\,\delta(x)\,\delta(y)\,\delta(z-vt)\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n p_0^2 v^2 \sin^4\theta}{4\pi^2 c^3}\left|\int \delta(x)\,\delta(y)\,\delta(z-vt)\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n p_0^2 v^2 \sin^4\theta}{4\pi^2 c^3}\left|\int e^{i\omega t(1-(nv/c)\cos\theta)}\,dt\right|^2,
\end{aligned} \tag{8}$$

where the third line follows from the second via integration by parts with respect to volume, and we take θ as the angle between $\hat{\mathbf{n}}$ and the z-axis.

The remaining integral is the same as in an intermediate step of the computation for Čerenkov radiation by electric charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, as on the top of p. 251 of [3], where the prefactor is $e^2\omega^2 n v^2 \sin^2\theta/4\pi^2 c^3$. Hence,

$$\frac{dU_\omega/d\Omega|_{\text{electric dipole}}}{dU_\omega/d\Omega|_{\text{charge } e}} = \frac{k^2 p_0^2 \sin^2\theta}{e^2} < \frac{p_0^2}{e^2 \lambda\!\!\!^{-2}}, \tag{9}$$

where the Čerenkov angle θ is related by $\cos\theta = c/nv$.

The electric dipole $\mathbf{p}_0$ might be that of an atom, in which case $p_0 \approx eR_{\text{Bohr}}$, where the Bohr radius is $R_{\text{Bohr}} \approx 5\times 10^{-11}$ m, and the ratio (9) would be $\approx (R_{\text{Bohr}}/\lambda\!\!\!^-)^2 \approx 2\times 10^{-7}$ for $\lambda = 600$ nm, *i.e.*, $\lambda\!\!\!^- \approx 10^{-7}$ m. While the Čerenkov radiation by such an electric dipole is strong compare to that of a neutron, it is very weak compared to that of an electron.

A Appendix: Charge and Current Densities for Point Dipoles $\mathbf{p}_0$ and $\mathbf{m}_0$

As reviewed in [9], the Lorentz transformation of electric and magnetic polarization densities, $\mathbf{P}$ and $\mathbf{M}$, from their rest frame (the $\star$ frame) to a frame in which they have velocity $\mathbf{v}$ is,

$$\mathbf{P} = \gamma\left(\mathbf{P}^\star + \frac{\mathbf{v}}{c}\times\mathbf{M}^\star\right) - (\gamma-1)(\hat{\mathbf{v}}\cdot\mathbf{P}^\star)\,\hat{\mathbf{v}}, \tag{10}$$

$$\mathbf{M} = \gamma\left(\mathbf{M}^\star - \frac{\mathbf{v}}{c}\times\mathbf{P}^\star\right) - (\gamma-1)(\hat{\mathbf{v}}\cdot\mathbf{P}^\star)\,\hat{\mathbf{v}}, \tag{11}$$

where c is the speed of light and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

For a "point" particle at the origin in its rest frame, with rest-frame electric and magnetic dipole moments $\mathbf{p}_0$ and $\mathbf{m}_0$, we write its electric and magnetic polarization densities as,

$$\mathbf{P}^\star = \mathbf{p}_0\, \delta^3\mathbf{r}^\star, \qquad \text{and} \qquad \mathbf{M}^\star = \mathbf{m}_0\, \delta^3\mathbf{r}^\star, \tag{12}$$

and the associated rest-frame charge and current densities as,

$$\rho^\star = -\boldsymbol{\nabla}^\star \cdot \mathbf{P}^\star = -(\mathbf{p}_0 \cdot \boldsymbol{\nabla}^\star)\, \delta^3\mathbf{r}^\star, \qquad \text{and} \qquad \mathbf{J}^\star = c\boldsymbol{\nabla}^\star \times \mathbf{M}^\star = -c\,\mathbf{m}_0 \times \boldsymbol{\nabla}^\star\, \delta^3\mathbf{r}^\star. \tag{13}$$

In a frame where the particle has velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, the charge and current densities are,

$$\rho = \gamma\left(\rho^\star + (\mathbf{J}^\star \cdot \hat{\mathbf{v}})\,\frac{\mathbf{v}}{c^2}\right), \qquad \mathbf{J} = \mathbf{J}^\star + (\gamma - 1)(\mathbf{J}^\star \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma\rho^\star\mathbf{v}. \tag{14}$$

We note that the Lorentz transformation of the 4-gradient $\partial_\mu = (\partial_t/c, -\boldsymbol{\nabla})$ tells us that $-\partial^\star_x = -\partial_x$, $-\partial^\star_y = -\partial_y$ and $-\partial^\star_z = \gamma(-\partial_z - (v/c)\partial_t/c)$, *i.e.*,

$$\boldsymbol{\nabla}^\star = \boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\,. \tag{15}$$

Hence, eqs. (13) and (14) combine to give,

$$\begin{aligned}
\rho = \gamma\Bigg(&-\mathbf{p}_0 \cdot \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(\gamma(z - vt)) \\
&-\frac{\mathbf{v}}{c}\hat{\mathbf{v}} \cdot \mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(\gamma(z - vt))\Bigg) \\
&= -\mathbf{p}_0 \cdot \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(z - vt) \\
&\quad -\frac{\mathbf{v}}{c}\hat{\mathbf{v}} \cdot \mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(z - vt),
\end{aligned} \tag{16}$$

and,

$$\begin{aligned}
\mathbf{J} = &-c\,\mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(\gamma(z - vt)) \\
&-(\gamma - 1)c\,\hat{\mathbf{v}}\left(\hat{\mathbf{v}} \cdot \mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(\gamma(z - vt))\right) \\
&-\gamma\mathbf{v}\left(\mathbf{p}_0 \cdot \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(\gamma(z - vt))\right) \\
&= -c\,\mathbf{m}_0 \times \left[\frac{\boldsymbol{\nabla}}{\gamma} + \frac{\gamma - 1}{\gamma}\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(z - vt) \\
&\quad -\frac{\gamma - 1}{\gamma}c\,\hat{\mathbf{v}}\left(\hat{\mathbf{v}} \cdot \mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(z - vt)\right) \\
&\quad -\mathbf{v}\left(\mathbf{p}_0 \cdot \left[\boldsymbol{\nabla} + (\gamma - 1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\delta(x)\,\delta(y)\,\delta(z - vt)\right).
\end{aligned} \tag{17}$$

These expressions are somewhat simpler for the special cases that $\mathbf{p}_0 \perp \mathbf{v}$ and $\mathbf{m}_0 \parallel \mathbf{v}$,

$$\rho = -\left((\mathbf{p}_0 \cdot \boldsymbol{\nabla}) + \frac{\mathbf{v}}{c} \, \hat{\mathbf{v}} \cdot \mathbf{m}_0 \times \boldsymbol{\nabla} \right) \delta(x)\, \delta(y)\, \delta(z - vt), \tag{18}$$

$$\mathbf{J} = -\left(\mathbf{v}\, (\mathbf{p}_0 \cdot \boldsymbol{\nabla}) + c\, \mathbf{m}_0 \times \boldsymbol{\nabla} \right) \delta(x)\, \delta(y)\, \delta(z - vt). \tag{19}$$

Equation (4) then follows from eq. (19) when $\mathbf{p}_0 = 0$, and eq. (7) follows when $\mathbf{m}_0 = 0$.

References

[1] L.I. Schiff, *Quantum Mechanics*, (McGraw-Hill, 1949),
http://kirkmcd.princeton.edu/examples/QM/schiff_qm_49.pdf

[2] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[3] K.T. McDonald, *Ph501 Lecture 21*,
http://kirkmcd.princeton.edu/examples/ph501/ph501lecture21.pdf

[4] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999),
http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[5] K.T. McDonald, *Ph501 Lecture 15*,
http://kirkmcd.princeton.edu/examples/ph501/ph501lecture15.pdf

[6] G.N. Afanasiev and Y.P. Stepanovsky, *Electromagnetic Fields of Electric, Magnetic, and Toroidal Dipoles Moving in Medium*, Phys. Scripta **61**, 704 (2000),
http://kirkmcd.princeton.edu/examples/EM/afanasiev_ps_61_704_00.pdf

[7] U. Leonhardt and Y. Rosenberg, *Cherenkov radiation of light bullets*, Phys. Rev. A **100**, 063802 (2019), http://kirkmcd.princeton.edu/examples/EM/leonhardt_pra_100_063802_19.pdf

[8] K.D. Bonin *et al.*, *Observation of Interference between Čerenkov and Synchrotron Radiation*, Phys. Rev. Lett. **57**, 2264 (1986),
http://kirkmcd.princeton.edu/examples/EM/bonin_prl_57_2264_86.pdf

[9] V. Hnizdo and K.T. McDonald, *Fields and Moments of a Moving Electric Dipole* (Nov. 29, 2011), http://kirkmcd.princeton.edu/examples/movingdipole.pdf