Falling Chimney

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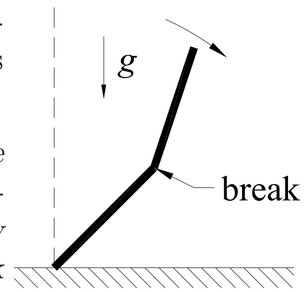
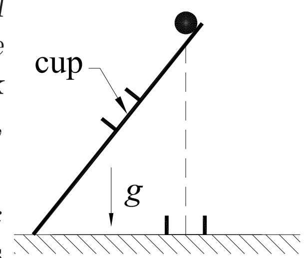
1 Problem

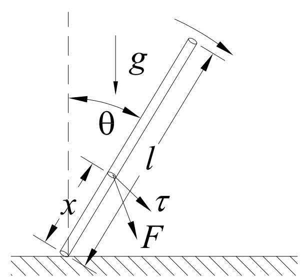
If a chimney is undermined on one side, so that it falls, rotating about its base, it usually snaps before hitting the ground. We can estimate the most likely position of the break by an extension of the principles of statics to a dynamic situation. This is the spirit of D'Alembert.



You might wish to convince yourself that the above picture shows the behavior after the break by performing a home experiment. A ball rests on the one end of a stick held initially at some angle to the horizontal, with the other end of the stick on the floor. Let the system loose. The stick will appear to fall faster that the ball. A cup placed on the stick can catch the ball after the stick hits the floor. Hence, the top end of the stick falls with acceleration greater than 1 g, and if the stick is weak, it will snap in the sense shown in the first figure.



Consider the lower portion of the chimney below a distance x from its base. The internal forces acting the lower portion across a slice of the chimney at x can be combined into a net force $\mathbf{F}$ applied at the center of the slice, and a net torque $\boldsymbol{\tau}$ acting about the center of the slice — a principle of statics. "Clearly" $\boldsymbol{\tau}$ is perpendicular to the vertical plane of the falling chimney. The torque $\boldsymbol{\tau}$ is due to pairs $\pm\mathbf{F}'$ of forces along the slice, such that this torque is the same when computed about any point along the centerline of the chimney between 0 and x. With respect to points on the centerline of the portion of the chimney from x to l, the force and torque on the slice are $-\mathbf{F}$ and $-\boldsymbol{\tau}$.



The chimney might break at x for any of 3 reasons:

1. The tension $F_{\|}$ along the chimney is too great for the mortar between the bricks to sustain. However, $F_{\|}$ is compressive in the case of the falling chimney, and cannot lead to a break.

2. The shear $F_{\perp}$ across the slice is too great.

3. The torque $\boldsymbol{\tau}$ is too great and the chimney bends and snaps.

Show that for an unbroken, falling chimney (of mass m, length l, with uniform, linear mass density m/l, and radius small compared to l) at angle θ to the vertical,

$$\tau(x) = \frac{mgx(l-x)^2\sin\theta}{4l^2}, \qquad \text{and} \qquad F_{\perp} = \frac{mgx(l-x)(l-3x)\sin\theta}{4l^2}, \tag{1}$$

such that the chimney most likely breaks at $x = l/3$ if torque matters, but at $x = 2l/3$ if shear matters. *We take $\boldsymbol{\tau}$ to be positive when out of the page.*

Empirically, most chimneys break near $x = 1/3$, suggesting that they break due to the torque effect.

Hint: Consider torque analyses of the entire chimney, and of the two portions described above.

https://www.youtube.com/watch?v=jI0ryk39H4w

2 Solution

The literature on the falling chimney includes [1]-[10].

The torque equation for the entire (unbroken) chimney about its base is,

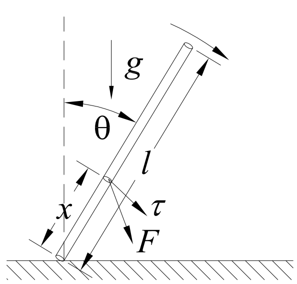
$$\frac{ml^2}{3}\ddot{\theta} = mg\frac{l}{2}\sin\theta, \qquad \ddot{\theta} = \frac{3g\sin\theta}{2l}, \tag{2}$$

noting that $\mathbf{F}$ and $\boldsymbol{\tau}$ are zero at the top of the chimney, and supposing the radius of the chimney is small compared to its length l

We next consider the torque equation for lower portion of the chimney from 0 to x, again taking the point of reference as the base of the chimney,

$$m\frac{x}{l}\frac{x^2}{3}\ddot{\theta} = m\frac{x}{l}g\frac{x}{2}\sin\theta + xF_\perp - \tau, \qquad \tau = xF_\perp + \frac{mgx^2(l-x)\sin\theta}{2l^2}, \tag{3}$$

using eq. (2) to obtain the second form of eq. (3).



We also consider the torque equation for the upper portion of the chimney from x to l (before it breaks). All points on this segment are accelerating, so it is perhaps best to use its center of mass as the reference point. Noting that $\mathbf{F}$ and $\boldsymbol{\tau}$ at x on the lower end of the upper segment are equal and opposite to those on the upper end of the lower segment, we have,

$$m\frac{l-x}{l}\frac{(l-x)^2}{12}\ddot{\theta} = \frac{l-x}{2}F_\perp + \tau, \qquad \frac{mg(l-x)^3\sin\theta}{8l^2} = F_\perp\frac{l-x}{2} + \tau, \tag{4}$$

recalling eq. (2). Using eq. (3) in (4) we obtain,

$$\begin{aligned} \frac{l+x}{2}F_\perp &= \frac{mg(l-x)^3\sin\theta}{8l^2} - \frac{mgx^2(l-x)\sin\theta}{4l^2} \\ \frac{mg(l-x)\sin\theta}{8l^2}[(l-x)^2 - 3x^2] &= \frac{mg(l+x)(l-x)(l-3x)\sin\theta}{8l}, \qquad (5) \\ F_\perp(x) &= \frac{mg(l-x)(l-3x)\sin\theta}{4l^2}. \qquad (6) \end{aligned}$$

Then, using eq. (3),

$$\tau(x) = \frac{mgx(l-x)(l-3x)\sin\theta}{4l^2} + \frac{mgx^2(l-x)\sin\theta}{2l^2} = \frac{mgx(l-x)^2\sin\theta}{4l^2}. \tag{7}$$

$F_\perp(x)$ is maximum at $x = 2l/3$, while $\tau(x)$ is maximum at $x = l/3$.

A Appendix: Torque Analyses about Other Points

The torque analysis for the upper portion of the chimney could also be carried out using the base of the chimney, or point x, or the upper end of the chimney (amongs other points).

A.1 Using the Base of the Chimney as the Reference Point

The moment of inertia of the upper portion of the chimney about its (fixed) base (point B) is

$$I_B = \int_x^l \frac{m}{l}x'^2\,dx' = \frac{m}{3l}(l^3 - x^3) = \frac{m(l-x)(l^2+lx+x^2)}{3l}. \tag{8}$$

Recalling that the force and torque acting at point x on the upper portion of the chimney are $-\mathbf{F}$ and $-\boldsymbol{\tau}$, the torque equation for the upper portion is

$$I_B\,\ddot{\theta} = -xF_\perp + \tau + \frac{m(l-x)}{l}g\left(x + \frac{l-x}{2}\right)\sin\theta. \tag{9}$$

With eqs. (2) and (8) this becomes

$$\tau = xF_\perp + \frac{mg(l-x)(l^2+lx+x^2)\sin\theta}{2l^2} - \frac{mg(l-x)(l+x)\sin\theta}{2l} = xF_\perp + \frac{mgx^2(l-x)\sin\theta}{2l^2}. \tag{10}$$

That is, torque analyses of both the lower and upper portions of the chimney using the base of the chimney as the reference point give the same result for the magnitude τ of the torque at point x.

A.2 Using Point x as the Reference Point

The moment of inertia of the upper portion of the chimney about point x is

$$I_x = m\frac{l-x}{l}\frac{(l-x)^2}{3} = \frac{m(l-x)^3}{3l}\,. \tag{11}$$

Point x has acceleration $\mathbf{a}_x = -x\dot{\theta}^2\,\hat{\mathbf{x}} + x\ddot{\theta}\,\hat{\boldsymbol{\theta}}$, so an observer at point x considers there to be a"fictitious" force $-m_i\,\mathbf{a}_x$ acting on each mass m_i in the upper portion of the chimney. See, for example, eq. (39.7) of [12] and pp. 168-172 of [13]. Since the "fictitious" force $-m_i\mathbf{a}_x$ that acts on mass m_i does not depend on the position r_i of that mass, the sum of the associated "fictitious" torques about point x on an object of total mass m is simply

$$\sum \mathbf{r}_i \times (-m_i\mathbf{a}_x) = \mathbf{r}_{\rm cm,upper} \times (-m_{\rm upper}\mathbf{a}_x), \tag{12}$$

with magnitude $[(l-x)/2][m(l-x)/l]x\,\ddot{\theta}$.

Recalling that the torque acting at point x on the upper portion of the chimney is $-\boldsymbol{\tau}$ (and that $\boldsymbol{\tau}$ is positive when out of the page), the torque equation for the upper portion is

$$\begin{aligned} I_x\,\ddot{\theta} &= \tau + |\mathbf{r}_{\rm cm,upper} \times m_{\rm upper}\mathbf{g}| - |\mathbf{r}_{\rm cm,upper} \times (-m_{\rm upper}\mathbf{a}_x)| \\ &= \tau + \frac{l-x}{2}\frac{m(l-x)}{l}g\sin\theta - \frac{l-x}{2}\frac{m(l-x)}{l}x\ddot{\theta} \end{aligned} \tag{13}$$

With eqs. (2), (3) and (11) this becomes

$$\begin{aligned} xF_\perp &= -\frac{mgx^2(l-x)\sin\theta}{2l^2} + \frac{mg(l-x)^3\sin\theta}{2l^2} - \frac{mg(l-x)^2\sin\theta}{2l} + \frac{3mgx(l-x)^2\sin\theta}{4l^2} \\ &= \frac{mg(l-x)\sin\theta}{4l^2}\left[-2x^2 + 2(l-x)^2 - 2l(l-x) + 3x(l-x)\right] = \frac{mgx(l-x)(l-3x)\sin\theta}{4l^2}\,, \end{aligned} \tag{14}$$

which agrees with eq. (6) for $F_\perp(x)$. Then, eq. (7) for $\tau(x)$ follows as before.

A.2.1 Use of an Accelerated and Rotating Frame

We can suppose that the (accelerated) point x is associated with a rotating coordinate system (with origin at x) with an arbitrary angular velocity $\boldsymbol{\Omega}(t)$ with respect to the lab frame, which requires consideration of the additional "fictitious" forces $m_i\mathbf{r}_i\times\dot{\boldsymbol{\Omega}}+2m_i\mathbf{v}_i\times\boldsymbol{\Omega}+m_i\boldsymbol{\Omega}\times(\mathbf{r}\times\boldsymbol{\Omega})$ that would act on mass m_i at $\mathbf{r}_i$ in the accelerated, rotating frame of point x, with velocity $\mathbf{v}_i$ in this frame). See, for example, eq. (39.7) of [12] and pp. 168-172 of [13].

When we only consider these "fictitious" forces, and sum over masses m_i in some object, the subscript i can be replaced by the subscript cm (for center of mass of the object). However, if we consider "fictitious" torques associated with these "fictitious" forces, then we must evaluate the sums over masses m_i. In general, it is expedient to avoid this additional effort by not using a rotating frame.

Nonetheless, we now suppose that the angular velocity Ω of the accelerated, rotating frame (with origin at point x) is $\dot{\theta}$, such that this frame is the body frame of the chimney, in which it is at rest.

The total torque about point x on the upper portion of the chimney is zero in this frame, which is the sum of the torque $-\boldsymbol{\tau}$ from the lower portion of the chimney, the torque due to gravity, and the "fictitious" torques due to the "fictitious" forces $-m_i\,\mathbf{a}_x$ and $m_i\mathbf{r}_i \times \dot{\boldsymbol{\Omega}}$ that act masses m_i in the upper portion.[1] The latter "fictitious" torque (about point x) is negative, with magnitude given by

$$\left|\sum \mathbf{r}_i \times (m_i\mathbf{r}_i \times \dot{\boldsymbol{\Omega}})\right| = \int_0^{l-x} \frac{m\,dr}{l}\,r^2\,\ddot{\theta} = \frac{m(l-x)^3}{3}\,\ddot{\theta} = I_x\,\ddot{\theta}. \tag{15}$$

Then, the torque equation in the accelerated, rotating frame about point x is, recalling eq. (12),

$$\begin{aligned} 0 &= \tau + |\mathbf{r}_{\rm cm,upper} \times m_{\rm upper}\mathbf{g}| - |\mathbf{r}_{\rm cm,upper} \times (-m_{\rm upper}\mathbf{a}_x)| - \left|\sum \mathbf{r}_i \times (m_i\mathbf{r}_i \times \dot{\boldsymbol{\Omega}})\right| \\ &= \tau + \frac{l-x}{2}\frac{m(l-x)}{l} g\sin\theta - \frac{l-x}{2}\frac{m(l-x)}{l} x\ddot{\theta} - I_x\,\ddot{\theta}, \end{aligned} \tag{16}$$

which is the same as eq. (13), but more laborious to deduce using the rotating frame.

We succeeded in using accelerated, rotating axes in the torque analysis for the special case that the axes are the body axes, considering the "fictitious" torques asscociated with the four types of "fictitious" forces in such frames of reference. However, it seems that for any other rotating axes there must be additional "fictitious" torques, such that the torque analysis reduces, in effect, to use of nonrotating axes.

This reinforces the well known advice not to use rotating axes in torque analyses.

A.3 Using the Top of the Chimney at the Reference Point

The moment of inertia of the upper portion of the chimney its top (point T) is the same as that about point x.

$$I_T = I_x = \frac{m(l-x)^3}{3l}\,. \tag{17}$$

Point T has acceleration $\mathbf{a}_T = -l\dot{\theta}^2\,\hat{\mathbf{x}} + l\ddot{\theta}\,\hat{\boldsymbol{\theta}}$, so an observer at point T considers there to be a "fictitious" force $-m_{\rm upper}\,\mathbf{a}_T$ acting on the center of the upper portion of the chimney. Recalling that the force torque acting at point x on the upper portion of the chimney are $-\mathbf{F}$ and $-\boldsymbol{\tau}$ (and that $\boldsymbol{\tau}$ is positive when out of the page), the torque equation for the upper portion is

$$I_T\,\ddot{\theta} = (l-x)F_\perp + \tau - \frac{m(l-x)}{l} g\frac{l-x}{2}\sin\theta + \frac{m(l-x)}{l} l\ddot{\theta}\frac{l-x}{2} \tag{18}$$

With eqs. (2), (3) and (17) this becomes

$$\begin{aligned} lF_\perp &= -\frac{mgx^2(l-x)\sin\theta}{2l^2} + \frac{mg(l-x)^3\sin\theta}{2l^2} + \frac{mg(l-x)^2\sin\theta}{2l} - \frac{3mgl(l-x)^2\sin\theta}{4l^2} \\ &= \frac{mg(l-x)\sin\theta}{4l^2}\left[-2x^2 + 2(l-x)^2 + 2l(l-x) - 3l(l-x)\right] = \frac{mgl(l-x)(l-3x)\sin\theta}{4l^2}, \end{aligned} \tag{19}$$

which agrees with eq. (6) for $F_\perp(x)$.

[1] The Coriolis force is zero in this frame, and the centrifugal force is along the chimney, producing no centrifugal torque.

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