An Atmospheric-Density Anomaly with a Magnetostatic Analog

Nathaniel E. Tarshish
Atmospheric and Oceanic Sciences, Princeton University, Princeton, NJ 08544
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Solar heating of a localized region of dust particles or water droplets can lead to a localized anomaly of density and pressure in the Earth's atmosphere.[1] Consider an idealized model of such an anomaly, with initial ellipsoidal shape, in an incompressible fluid with density $\rho(\mathbf{r},t) = \rho_0 + \rho_1(\mathbf{r},t)$ and pressure $p(\mathbf{r},t) = p_0(z) + p_1(\mathbf{r},t)$, where ρ_0 is constant over the volume relevant to the evolution of the anomaly, and $p_0(z)$ is the pressure in the absence of the anomaly. Initially the velocity field $\mathbf{u}(\mathbf{r}, t = 0)$ is everywhere zero over this volume, and the initial density perturbation $\rho_1(t = 0)$ is uniform over the (initial) ellipsoidal volume of the anomaly (and zero outside this volume).

Discuss the early-time evolution of the anomaly, assuming that the perturbations are always small, and that the volume of the anomaly is small enough that the gravitational acceleration $-g\,\hat{\mathbf{z}}$ can be regarded as constant. Viscosity, as well as electric, magnetic and thermal forces on the system, can be ignored.[2]

2 Solution

The equation of motion of an incompressible ($\boldsymbol{\nabla}\cdot\mathbf{u} = 0$), inviscid fluid subject only to the "external" force of gravity was given by Euler [6],

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right) = -\boldsymbol{\nabla} p - \rho\, g\,\hat{\mathbf{z}}. \tag{1}$$

We write the pressure as $p(\mathbf{r},t) = p_0(z) + p_1(\mathbf{r},t)$, where the unperturbed pressure $p_0(z)$ varies only with height z and obeys,

$$\boldsymbol{\nabla} p_0 = \frac{dp_0}{dz} = -\rho_0\, g\,\hat{\mathbf{z}}, \tag{2}$$

such that in the absence of the density anomaly the system would remain at rest (assuming it to be initially at rest).

We emphasize the perturbed system at early times, when the velocity field $\mathbf{u}$ is still small, and we neglect the squares and products of small quantities, such as $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ and $\rho_1\,\partial\mathbf{u}/\partial t$.[3]

[1] For reviews, see, for example, [1, 2]. A hot-air balloon is such an anomaly. Less localized anomalies exist in the ionosphere, such as that reported in [3]. See also [4].

[2] This problem is also discussed in [5].

[3] This approximation is an extension of that associated with Boussinesq [7].
https://en.wikipedia.org/wiki/Boussinesq_approximation_(buoyancy)

Then, Euler's equation (1) simplifies to,

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} p_1 - \rho_1\, g\, \hat{\mathbf{z}}. \tag{3}$$

On taking the divergence of eq. (3), we have,

$$\nabla^2 p_1 = -g \frac{\partial \rho_1}{\partial z} = 4\pi \left(-\frac{g}{4\pi} \frac{\partial \rho_1}{\partial z} \right). \tag{4}$$

We now make the further approximation that at time $t = 0$ the density perturbation ρ_1 is uniform over an ellipsoidal volume, and zero outside this. Then, the derivative $\partial \rho_1(t = 0)/\partial z$ corresponds to the result of the superposition of a volume (ellipsoid) of density ρ_1/dz with a volume (ellipsoid) of density $-\rho_1/dz$ where the latter volume is offset by dz in z from the former. This is equivalent to surface density $\rho_1\, \hat{\mathbf{z}} \cdot \hat{\mathbf{n}}$ on the volume of the anomaly, where $\hat{\mathbf{n}}$ is the outward, normal unit vector from this volume. In electrostatics and magnetostatics, such a surface density is familiar as the result of a uniform volume density, $\rho_1\, \hat{\mathbf{z}}$, of dipoles. Hence, according to eq. (4), the pressure anomaly p_1 is equivalent to the scalar potential V due to a permanent magnet in the shape of the density anomaly, with uniform magnetization density (in Gaussian units),

$$\mathbf{M} = -\frac{\rho_1\, g}{4\pi}\, \hat{\mathbf{z}}, \tag{5}$$

and $-\boldsymbol{\nabla} p_1$ is equivalent to the corresponding magnetic field $\mathbf{H} = -\boldsymbol{\nabla} V$.

As discussed by Maxwell in Arts. 437-438 of [8][4] (who followed Poisson [9], Neumann [10] and Thomson, p. 471 of [11]), the field $\mathbf{H}$ is uniform inside an ellipsoid with uniform magnetization density $\mathbf{M}$ (although the interior $\mathbf{H}$ field is not in the same direction as the magnetization $\mathbf{M}$ unless the latter is parallel to one of the axes of the ellipsoid). As such, the righthand side of eq. (3) inside the anomaly is a constant vector, and the anomaly accelerates, keeping its shape, for early times.[5]

We now restrict our attention to the case of a spheroidal anomaly, initially centered on the origin, with its symmetry axis along z, initial radius a in the x-y plane, and initial half-height b in z. Then, $-\boldsymbol{\nabla} p_1$ (and the equivalent $\mathbf{H}$) will be parallel to the equivalent $\mathbf{M}$ of eq. (5), *i.e.*, to $\hat{\mathbf{z}}$. *The case when the symmetry axis is not along z will be discussed in sec. 2.4.*

2.1 Spherical Anomaly

For a spherical anomaly, with $a = b$, the internal magnetic field is (in Gaussian units and at early times),

[4]In Maxwell's notation, $\mathbf{M} = (A, B, C) = I(l, m, n)$, where $I = |\mathbf{M}|$ and (l, m, n) are the direction cosines of $\mathbf{M}$, supposing that the axes of the ellipsoid are parallel to (x, y, z). The internal magnetic field is $\mathbf{H} = -(AL, BM, CN)$ where (L, M, N) are given for special cases in eqs. (10)-(15), Art. 438 of [8]. The quantities (L, M, N) used to be called the demagnetization factors, perhaps following [12], although since [13] it is usual to denote $(N_x, N_y, N_z) = (L, M, N)/4\pi$, which obey $N_x + N_y + N_z = 1$, as the demagnetizing factors. See also secs. 4, 8 and 29 of [14].

[5]A different analogy between fluid dynamics and magnetostatics was explored by Thomson, who claimed to follow Euler, on pp. 444-450 and pp. 579-587 of [11].

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = -\frac{4\pi\mathbf{M}}{3} \qquad \text{(interior)}, \tag{6}$$

according to eq. (15) of Art. 438 of [8].[6] Then, recalling eq. (5),

$$-\boldsymbol{\nabla} p_1 = \frac{\rho_1\, g}{3}\,\hat{\mathbf{z}}, \tag{7}$$

and eq. (3) for the acceleration of the velocity field inside the sphere becomes (for early times),[7]

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0} - \frac{\rho_1}{\rho_0}\, g\,\hat{\mathbf{z}} = -\frac{2\rho_1}{3\rho_0}\, g\,\hat{\mathbf{z}} \qquad \text{(interior)}. \tag{8}$$

That is, (if $\rho_1 > 0$) the anomaly falls with constant acceleration, and maintains its shape (for times $\lesssim \sqrt{a/g}$), followed by turbulent motion.[8]

Outside the spherical anomaly, the acceleration field at early times has the form of a dipole field,

$$\frac{\partial \mathbf{u}(\mathbf{r})}{\partial t} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0} = -\frac{a^3 \rho_1\, g}{3\rho_0}\frac{3(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})\,\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} \qquad \text{(exterior)}, \tag{9}$$

where $\mathbf{r}$ is measured from the center of the anomaly.

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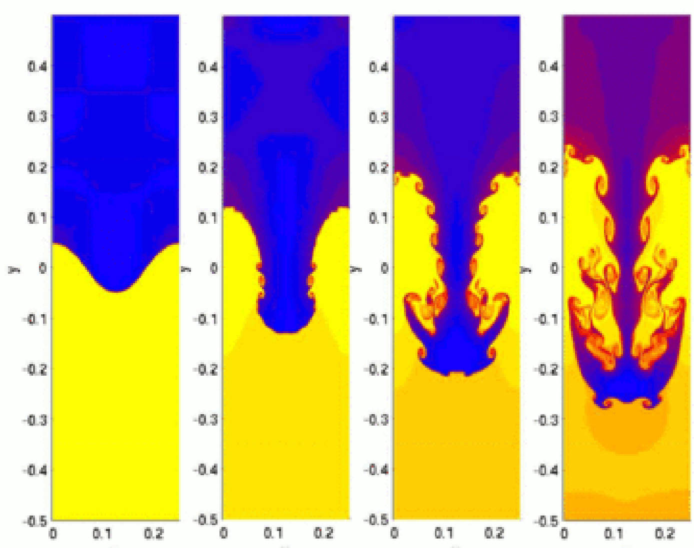
[6] A more straightforward derivation of this result was given by W. Thomson on pp. 470-471 of [11] (who attributed aspects of the argument to Poisson on p. 362). See also sec. 2.1 of [15].

[7] The quantity $\rho_1\, g/\rho_0$ is sometimes called the buoyancy.

[8] This result is somewhat related to the Rayleigh-Taylor instability [16, 17], which involves a simple acceleration of one viscous fluid through another at early times, followed by later, turbulent behavior (governed by the Navier-Stokes equation [18, 19]), as shown in the figure below (from https://en.wikipedia.org/wiki/Rayleigh-Taylor_instability)



2.2 Oblate Spheroid ("Pancake")

For an oblate spheroid with $R = b/a < 1$, the internal magnetic field is, for $\mathbf{M}$ parallel to the symmetry axis z,

$$\mathbf{H} = -4\pi k\,\mathbf{M} \qquad \text{with} \qquad k = \frac{1}{1-R^2} - \frac{R\sin^{-1}\sqrt{1-R^2}}{(1-R^2)^{3/2}} \qquad k(0) = 1, \qquad k(1) = \frac{1}{3}\,, \tag{10}$$

according to eq. (12) of Art. 438 of [8].[9] A plot of $1 - k$ *vs.* R appears below.

Then, $-\boldsymbol{\nabla} p_1 = k\,\rho_1\,g\,\hat{\mathbf{z}}$, and eq. (3) for the acceleration of the velocity field inside the anomaly is (for early times),

$$\frac{\partial \mathbf{u}}{\partial t} = -(1-k)\frac{\rho_1}{\rho_0}\,g\,\hat{\mathbf{z}} \qquad \text{(interior of oblate spheroid).} \tag{11}$$

For a "pancake" anomaly, with $R \approx 0$ and $k \approx 1$, the acceleration field is negligible.[10] *Similarly, if the anomaly had the form of a short, right-circular cylinder of radius a and height $h \ll a$, then the equivalent internal field would be $\mathbf{H} \approx -4\pi\mathbf{M} \to \rho_1\,g\,\hat{\mathbf{z}}$, and the acceleration of the anomaly would also be zero for early times.*

For uniformly magnetized ellipsoids other than spheres, the external fields can be expressed in terms of elliptic integrals. Numerical approximations are given, for example, in [13, 21]-[23], and for other shapes in, for example, [24]-[28].

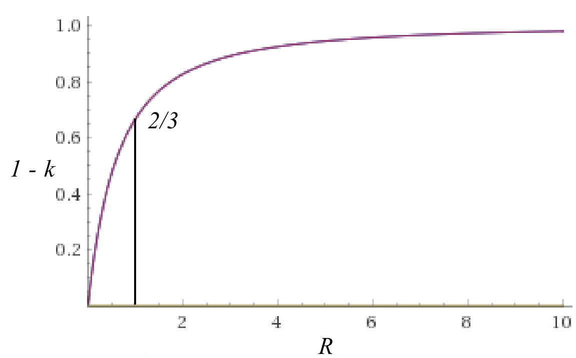
2.3 Prolate Spheroid ("Needle")

For a prolate spheroid with $R = b/a > 1$, the internal magnetic field is,

$$\mathbf{H} = -4\pi k\mathbf{M}, \qquad k = \frac{1}{R^2-1}\left(\frac{R\ln(R+\sqrt{R^2-1})}{\sqrt{R^2-1}} - 1\right), \qquad k(1) = \frac{1}{3}\,, \qquad k(\infty) = 0, \tag{12}$$

according to eq. (14) of Art. 438 of [8].[11] Again, the acceleration field inside the anomaly at early times has the form,

$$\frac{\partial \mathbf{u}}{\partial t} = -(1-k)\frac{\rho_1}{\rho_0}\,g\,\hat{\mathbf{z}} \qquad \text{(interior of spheroid).} \tag{13}$$



[9]Our a is Maxwell's $b = c$ of his eq. (11), Art. 438 of [8], our b is his a, his e is our $\sqrt{1-R^2}$. Our k, the demagnetizing factor associated with the symmetry axis, is $L/4\pi$ of his eq. (12).

[10]This result was anticipated in [20].

[11]Our a is Maxwell's $a = b$ of his eq. (13), Art. 438 of [8], our b is his c, his e is our $\sqrt{R^2-1}/R$, and N of his eq. (14) is our $4\pi k$.

For a "needle" anomaly ($R \to \infty,\ k \to 0$), $-\boldsymbol{\nabla} p_1 \approx 0$ for its interior, and eq. (3) for the acceleration of the velocity field inside the anomaly becomes (for early times),

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\rho_1}{\rho_0}\, g\, \hat{\mathbf{z}} \qquad (\text{"needle"}). \tag{14}$$

Similarly, if the anomaly had the form of a tall, right-circular cylinder of radius a and height $h \gg a$ (rod), then the equivalent $\mathbf{H}$ inside the anomaly would be nonzero near the top and bottom of the cylinder but would be very small over most of its length, in approximate agreement with the above analysis of a spheroidal "needle".

2.4 Nonvertical Spheroid

For completeness, we consider the case of a spheroidal anomaly whose symmetry axis makes angle α to the vertical in the x-z plane.

The equivalent magnetization density is still given by eq. (5), but the equivalent $\mathbf{H}$ field is not parallel to the z-axis. We introduce the rotated coordinates (x', y, z'), whose axes are parallel to the axes of the nonvertical spheroid. Then, the equivalent magnetization density can be written as,

$$\mathbf{M} = -\frac{\rho_1\, g}{4\pi}\, \hat{\mathbf{z}} = -\frac{\rho_1\, g}{4\pi}\, (-\hat{\mathbf{x}}' \sin\alpha + \hat{\mathbf{z}}' \cos\alpha), \tag{15}$$

and the equivalent $\mathbf{H}$ field inside the spheroid is given by,

$$\begin{aligned} -\boldsymbol{\nabla} p_1 = \mathbf{H} &= 4\pi \frac{1-k}{2} M_{x'}\, \hat{\mathbf{x}}' + 4\pi k M_{z'}\, \hat{\mathbf{z}}' \\ &= -\rho_1 g \left\{ \left[\frac{3k-1}{2} \sin\alpha\cos\alpha \right] \hat{\mathbf{x}} + \left[k + \frac{1-3k}{2} \sin^2\alpha \right] \hat{\mathbf{z}} \right\}, \end{aligned} \tag{16}$$

recalling that the demagnetizing factors $(N_{x'}, N_y, N_{z'})$ are given by $N_{z'} = k$ of eqs. (10) or (12), with $N_{x'} = N_y = (1-k)/2$, since the factors sum to unity. The acceleration field $\partial \mathbf{u}/\partial t$ is constant at early times inside the spheroid, recalling eq. (3),

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\boldsymbol{\nabla} p_1}{\rho_0} - \frac{\rho_1\, g}{\rho_0}\, \hat{\mathbf{z}} = -\frac{\rho_1\, g}{\rho_0} \left\{ \left[\frac{3k-1}{2} \sin\alpha\cos\alpha \right] \hat{\mathbf{x}} + \left[1 + k + \frac{1-3k}{2} \sin^2\alpha \right] \hat{\mathbf{z}} \right\}, \tag{17}$$

with magnitude,

$$\left| \frac{\partial \mathbf{u}}{\partial t} \right| = \frac{\rho_1\, g}{\rho_0} \sqrt{(1+k)^2 + \frac{5+k}{4}(1-3k)\sin^2\alpha}, \tag{18}$$

and with angle β to the vertical related by,

$$\tan\beta = \frac{(3k-1)\sin\alpha\cos\alpha}{2(1+k) + (1-3k)\sin^2\alpha}\,. \tag{19}$$

If the symmetry axis of the spheroidal anomaly were horizontal, then the interior acceleration field would again be vertical at early times,

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{3-k}{2} \frac{\rho_1\, g}{\rho_0}\, \hat{\mathbf{z}} \qquad (\alpha = \pi/2). \tag{20}$$

The contrails from airplanes might be an example of a horizontal "needle" anomaly ($k = 0$).[12]

[12] https://en.wikipedia.org/wiki/Contrail.

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