An Off-Center "Coaxial" Cable

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1 Problem

A "coaxial" transmission line has inner conductor of radius a and outer conductor of radius b, but the axes of these two cylinders are offset by a small distance $\delta \ll b$. Deduce the capacitance and inductance per unit length, and the impedance Z, accurate to order δ^2/b^2.

The (relative) dielectric constant and permeability of the medium between the two conductors both unity. The relevant frequencies and conductivities are so large that the skin depth is small compared to δ.

2 Solution

We use Gaussian units, and convert the impedance $Z = \sqrt{L/C}$ to MKSA units by noting that $1/c = 30\Omega$, where c is the speed of light.

We don't need to calculate both the capacitance C per unit length and the inductance L per unit length, since in the case of a (perfectly conducting) transmission line they are related by,

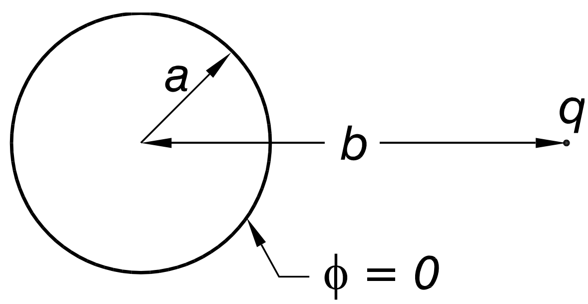
$$LC = \frac{\epsilon\mu}{c^2}, \tag{1}$$

where the dielectric constant ϵ and the permeability μ are unity in the present case. The assumed smallness of the skin depth permits us to approximate the present transmission line as perfectly conducting.

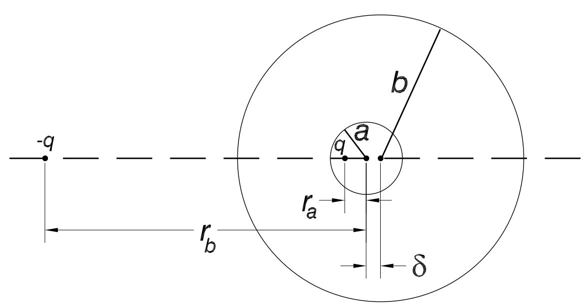
We first present two calculations of the capacitance (secs. 3 and 4), and then a calculation of the inductance (sec. 5) as illustrations of various possible techniques.

3 The Capacitance Via the Image Method

It is expedient to use the image method for 2-dimensional cylindrical geometries. Recall that in the case of a wire of charge q per unit length at distance b from a ground conducting cylinder of radius a, as shown in the figure, one can think of an image wire of charge $-q$ at radius a^2/b.



To apply this to the present problem, sketched in the figure below, note that the image wires of charge $\pm q$ per unit length are both located to the left of the center of the inner conductor, say at distances r_a and r_b.



For the inner cylinder to be an equipotential, we must have,

$$r_b = \frac{a^2}{r_b}, \tag{2}$$

and the outer cylinder is also an equipotential provided,

$$r_b + \delta = \frac{b^2}{r_a + \delta}, \tag{3}$$

noting the offset by δ between the inner and outer cylinder. Combining eqs. (2) and (3), and noting that $r_a \to 0$ as $\delta \to 0$, we find,

$$r_a = \frac{b^2 - a^2 - \delta^2 - \sqrt{(b^2 - a^2 - \delta^2)^2 - 4a^2\delta^2}}{2\delta}. \tag{4}$$

The capacitance is related by $C = q/\Delta V$, where $\Delta V = V_b - V_a$ is the potential difference between the two cylinders. Recall that the potential at distance r from a wire of charge q per unit length is $2q \ln r$ + constant. We evaluate the potentials at the points where the cylinders are closest to, one another:

$$V_a = 2q \ln(a - r_a) - 2q \ln(r_b - a) = 2q \ln \frac{a - r_a}{a^2/r_a - a} = 2q \ln \frac{r_a}{a}, \tag{5}$$

using eq. (2), and,

$$V_b = 2q \ln(b - \delta - r_a) - 2q \ln(r_b - b + \delta) = 2q \ln \frac{b - r_a - \delta}{b^2/(r_a + \delta) - b} = 2q \ln \frac{r_a + \delta}{b}, \tag{6}$$

using eq. (3). Then,

$$\Delta V = 2q \ln \left[\frac{a}{b} \left(1 + \frac{\delta}{r_a} \right) \right]. \tag{7}$$

When combined with eq. (4), this is an "exact" solution for any $\delta < b - a$. In particular, as $\delta \to b - a$, then $r_a \to a$, and the cylinders touch with the result that $\Delta V = 0$.

Here, we suppose that $\delta \ll b - a$, and expand δ/r_a to second order,

$$\frac{\delta}{r_a} = \frac{b^2 - a^2 - \delta^2 + \sqrt{(b^2 - a^2 - \delta^2)^2 - 4a^2\delta^2}}{2a^2} \approx \frac{b^2 - a^2}{a^2} - \frac{b^2\delta^2}{a^2(b^2 - a^2)}, \tag{8}$$

so that,

$$1 + \frac{\delta}{r_a} \approx \frac{b^2}{a^2} \left(1 - \frac{\delta^2}{b^2 - a^2} \right). \tag{9}$$

The capacitance per unit length is therefore,

$$C = \frac{q}{\Delta V} \approx \frac{1}{2\left(\ln \frac{b}{a} - \frac{\delta^2}{b^2 - a^2} \right)}, \tag{10}$$

using eq. (7).

The inductance per unit length now follows from eq. (1),

$$L = \frac{2}{c^2}\left(\ln\frac{b}{a} - \frac{\delta^2}{b^2 - a^2}\right), \tag{11}$$

and the impedance is,

$$Z = \sqrt{\frac{L}{C}} \approx \frac{2}{c}\left(\ln\frac{b}{a} - \frac{\delta^2}{b^2 - a^2}\right) = 60\left(\ln\frac{b}{a} - \frac{\delta^2}{b^2 - a^2}\right)\ \Omega. \tag{12}$$

Remark: The "exact" expression (7) is often written in a different fashion, which is convenient for large δ, but perhaps less useful for small δ. The "exact" version of (8) leads to,

$$1 + \frac{\delta}{r_a} = \frac{b^2 + a^2 - \delta^2 + \sqrt{(b^2 + a^2 - \delta^2)^2 - 4a^2b^2}}{2a^2}, \tag{13}$$

which in turn leads to,

$$C = \frac{q}{\Delta V} = \frac{1}{2\ln\frac{b^2+a^2-\delta^2+\sqrt{(b^2+a^2-\delta^2)^2-4a^2b^2}}{2ab}} = \frac{1}{2\cosh^{-1}\frac{a^2+b^2-\delta^2}{2ab}}. \tag{14}$$

4 Capacitance Via Series Expansion of the Potential

The image method can be deduced by an application of series expansion techniques for the electrostatic potential. In this section, we explore a direct use of such techniques. A full solution is long, and when we leave off some steps at the end, we get an answer that is not quite correct.

We define the electrostatic potential ϕ to be zero on the inner conductor,

$$\phi(r = a) = 0, \tag{15}$$

and V on the outer conductor whose surface is approximately given by $r = b + \delta\cos\theta$,

$$\phi(r = b + \delta\cos\theta) = V. \tag{16}$$

The potential is symmetric about $\theta = 0$,

$$\phi(-\theta) = \phi(\theta), \tag{17}$$

so terms in $\sin n\theta$ cannot appear in the series expansion of the potential,

$$\phi(r,\theta) = A_0 \ln r + \sum_{n=1}\left(A_n r^n + \frac{B_n}{r^n}\right)\cos n\theta. \tag{18}$$

The capacitance C per unit length is, of course, given by $C = Q/V$, where the charge Q per unit length on the inner conductor is given by,

$$Q = 2\pi a\int_0^{2\pi}\sigma(\theta)\ d\theta = 2\pi a\int_0^{2\pi}\frac{E_r(a,\theta)}{4\pi}\ d\theta = \frac{a}{2}\int_0^{2\pi}\frac{\partial\phi(a,\theta)}{\partial r}\ d\theta = \frac{A_0}{2}. \tag{19}$$

Thus,

$$C = \frac{A_0}{2V}. \tag{20}$$

Applying the boundary condition (15) to the general form (18), we have,

$$0 = A_0 \ln a + \sum_{n=1} \left(A_n a^n + \frac{B_n}{a^n} \right) \cos n\theta. \tag{21}$$

Likewise, the boundary condition (16) yields,

$$V = A_0 \ln(b + \delta \cos\theta) + \sum_{n=1} \left(A_n (b + \delta \cos\theta)^n + \frac{B_n}{(b + \delta\cos\theta)^n} \right) \cos n\theta. \tag{22}$$

With considerable effort, the terms in eq. (22) of the form $\cos^l \theta \cos m\theta$ can be expressed as sums of terms in the orthogonal set of functions $\cos n\theta$. Then, eqs. (21) and (22) can be combined to yield the Fourier coefficients A_n and B_n. Thus, subtracting eq. (21) from (22) and using the approximation (34), we have,

$$V = A_0 \left(\ln \frac{b}{a} + \frac{\delta \cos\theta}{b} - \frac{\delta^2 \cos^2\theta}{2b^2} \right) + F(A_n, B_n, \theta) \tag{23}$$

IF the integral of F with respect to θ vanished, then integrating eq. (23) yields,

$$V = A_0 \left(\ln \frac{b}{a} - \frac{\delta^2}{4b^2} \right), \tag{24}$$

and the capacitance would be,

$$C = \frac{A_0}{2V} \approx \frac{1}{2\left(\ln \frac{b}{a} - \frac{\delta^2}{4b^2} \right)}. \tag{25}$$

However, we the presence of terms like $A_1 \cos^2\theta$ in F means that we cannot expect its integral to vanish, and eq. (25) is not quite correct.

5 Calculation of the Inductance

The calculation of the inductance is complicated by the fact that the currents in this problem are distributed over surfaces, rather than flowing in filamentary wires. We would like to use the relation,

$$\Phi = cLI, \tag{26}$$

where I is the total (steady) current flowing down the inner conductor (and back up the outer conductor), and Φ is the magnetic flux per unit length linked by the circuit. From Ampère's law, with the assumption that the currents are uniformly distributed on the inner and outer conductors, the azimuthal component B_θ of the magnetic field in the region between the two conductors is given by,

$$B_\theta(r) = \frac{2I}{cr}. \tag{27}$$

If the cable were truly coaxial, the flux would be simply,

$$\Phi_0 = \int_a^b B_\theta dr = \frac{2I}{c} \ln \frac{b}{a}, \tag{28}$$

and the corresponding inductance would be,

$$L_0 = \frac{2}{c^2} \ln \frac{b}{a}. \tag{29}$$

Then, from eq. (1) the capacitance would be,

$$C_0 = \frac{1}{2 \ln(b/a)}, \tag{30}$$

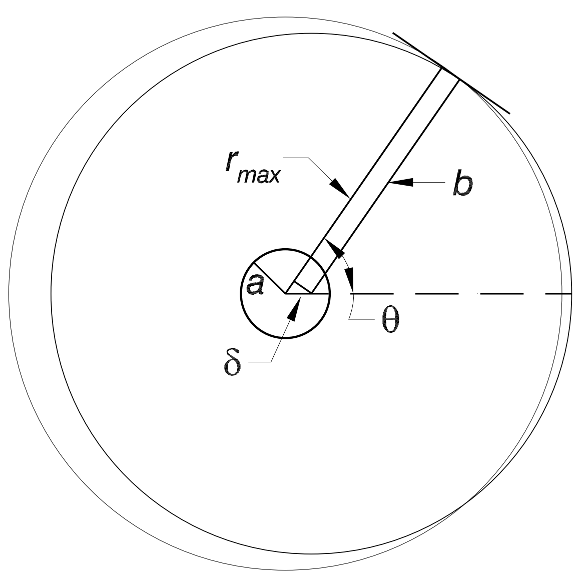
as is readily verified by an electrostatic analysis, and the transmission-line impedance would be,

$$Z_0 = \sqrt{\frac{L_0}{C_0}} = \frac{2}{c} \ln \frac{b}{a} = 60 \ \ln \frac{b}{a} \ \Omega. \tag{31}$$

However, because the outer conductor is off center with respect to the inner, we cannot simply use eq. (28). We can segment the currents on the conductors into filaments of azimuthal extent $d\theta$, and calculate the flux $\Phi(\theta)$ linked the circuit element defined by the segments centered on angle θ on the inner and outer conductors. Then, the effective inductance of the whole cable can be estimated from eq. (26) using the average of $\Phi(\theta)$,

$$L = \frac{1}{2\pi c I} \int_0^{2\pi} \Phi(\theta)\ d\theta = \frac{1}{2\pi c I} \int_0^{2\pi} d\theta \int_a^{r_{\max}(\theta)} B_\theta(r) dr = \frac{1}{\pi c^2} \int_0^{2\pi} \ln \frac{r_{\max}(\theta)}{a} d\theta, \tag{32}$$

using (27) and (28). The result holds only to the extent that the current distribution is independent of azimuth, as discussed in sec. 6. However, there will be a small azimuthal dependence to the current in this problem, so we will not obtain a completely correct result.



To complete the analysis, we need $r_{\rm max}(\theta)$, the maximum radius about the center of the inner conductor of magnetic field lines that are linked by the segment of the outer conductor at azimuthal angle θ. Assuming the currents is uniformly distributed over the inner and outer conductors, the magnetic field between the two conductors is entirely due to the current in the inner conductor, and the field is purely azimuthal about the axis of the inner conductor as given by eq. (27). Then, the geometry shown in the figure tells us that,

$$r_{\rm max}(\theta) = b + \delta\cos\theta. \tag{33}$$

This relation is "exact" to the extent that the currents are uniformly distributed; however, this is not actually the case in the present problem.

To use relation (33) in eq. (32), we approximate,

$$\ln\frac{r_{\rm max}(\theta)}{a} = \ln\frac{b+\delta\cos\theta}{a} = \ln\frac{b}{a} + \ln\left(1+\frac{\delta\cos\theta}{b}\right) \approx \ln\frac{b}{a} + \frac{\delta\cos\theta}{b} - \frac{\delta^2\cos^2\theta}{2b^2}, \tag{34}$$

which leads to,

$$L \approx \frac{2}{c^2}\left(\ln\frac{b}{a} - \frac{\delta^2}{4b^2}\right). \tag{35}$$

This result happens to agree with the result implied by sec. 4, but differs somewhat from the more accurate result of sec. 3.

6 The Magnetic Flux Linked by a Distributed Circuit

The magnetic flux through a filamentary circuit (one in which the conductors are idealized as wires) is well defined as,

$$\Phi = \int \mathbf{B}\cdot d\mathbf{S}, \tag{36}$$

where the integral is taken over any surface bounded by the circuit. However, when the conductors of the circuit are distributed and have a finite cross sectional area A, then eq. (36) is not well defined.

We wish to show that a consistent definition of the flux through a distributed circuit is obtained by segmenting the conductors into a large number of circuits each with very small cross sectional area A_i, and defining,

$$\Phi = \frac{1}{A}\sum_i A_i\Phi_i, \tag{37}$$

where the magnetic flux through subcircuit i is given by eq. (36).

We are interested in a definition of flux that gives consistency to the relation (26) in the context of circuit analysis. In particular, if the circuit has total resistance R, and the magnetic flux is changing, then we desire Faraday's law to be written as,

$$IR = \mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}, \tag{38}$$

which is the same form as holds for each of the filamentary subcircuits,

$$I_i R_i = \mathcal{E}_i = -\frac{1}{c}\frac{d\Phi_i}{dt}. \tag{39}$$

We suppose that the current flowing in subcircuit i is related to the total current according to,

$$I_i = \frac{A_i}{A} I, \tag{40}$$

in which case the resistance of subcircuit i is given by,

$$R_i = \frac{A}{A_i} R. \tag{41}$$

Then, we can combine eqs. (39)-(41) as,

$$I = \sum_i I_i = -\frac{1}{c}\sum_i \frac{1}{R_i}\frac{d\Phi_i}{dt} = -\frac{1}{cRA}\sum_i A_i \frac{d\Phi_i}{dt}. \tag{42}$$

Hence, the definition (37) leads to the desired relation (38) for the distributed circuit.