Skiing on a Cosine Hill

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1 Problem

In a variant of the famous problem of skiing/sliding on a cylindrical hill, consider a hill with surface $y = y_0 + A\cos(kx)$ (perhaps formed by glaciers). What is the largest value of k (for a given A) such that a skier who starts from rest at the top of the hill never leaves the (frictionless) surface while sliding down?

2 Solution

When the skier has traveled horizontal distance x from the top of the frictionless hill his/her speed is given by,

$$v^2 = 2g\Delta y = 2gA(1 - \cos kx), \tag{1}$$

where g is the acceleration due to gravity. At that position, the hill has angle $\theta > 0$ to the horizontal given by,

$$\tan\theta = |y'| = kA\,|\sin kx|\,, \qquad \sin\theta = \frac{|y'|}{1 + y'^2} = \frac{kA\,|\sin kx|}{1 + k^2A^2\sin^2 kx}\,, \tag{2}$$

and radius of curvature R,

$$R = \frac{(1 + y'^2)^{3/2}}{|y''|} = \frac{(1 + k^2A^2\sin^2 kx)^{3/2}}{k^2A\,|\cos kx|}\,. \tag{3}$$

When $\cos kx > 0$ the center of curvature is below the surface, and the normal component of Newton's equation of motion is,

$$mg\sin\theta - N = \frac{mv^2}{R} \qquad (\cos kx > 0), \tag{4}$$

where N is the normal force of the surface on the skier. If the skier loses contact with the hill at angle θ, then $N = 0$ and,

$$\sin\theta = \frac{kA\sin kx}{1 + k^2A^2\sin^2 kx} = \frac{v^2}{gR} = 2A(1 - \cos kx)\frac{k^2A\cos kx}{(1 + k^2A^2\sin^2 kx)^{3/2}}\,, \tag{5}$$

$$\sin kx(1 + k^2A^2\sin^2 kx)^{1/2} = 2kA(1 - \cos kx)\cos kx, \tag{6}$$

$$\sin^2 kx(1 + k^2A^2\sin^2 kx) = 4k^2A^2(1 - 2\cos kx + \cos^2 kx)\cos^2 kx, \tag{7}$$

$$(1 - \cos^2 kx)[1 + k^2 A^2(1 - \cos^2 kx)] = 4k^2 A^2(\cos^2 kx - 2\cos^3 kx + \cos^4 kx), \tag{8}$$

$$f(\cos kx) = 3k^2 A^2 \cos^4 kx - 8k^2 A^2 \cos^3 kx + (1 + 6k^2 A^2)\cos^2 kx - 1 - k^2 A^2 = 0. \tag{9}$$

We are interested in the special case that the quartic polynomial f barely has a real solution x_0, which implies that this solution is also at the minimum of f,

$$\begin{aligned} 0 = \frac{d\,f(\cos kx_0)}{d\cos kx} &= 12k^2 A^2 \cos^3 kx_0 - 24k^2 A^2 \cos^2 kx_0 + 2(1 + 6k^2 A^2)\cos kx_0 \\ &= 2\cos kx_0(6k^2 A^2 \cos^2 kx_0 - 12k^2 A^2 \cos kx_0 + 1 + 6k^2 A^2). \end{aligned} \tag{10}$$

The solution $\cos kx_0 = 0$ to eq. (10) is not a solution to the quartic equation (8), so if a minimum exists with $f = 0$ it must be that,

$$\cos^2 kx_0 - 2\cos kx_0 + 1 + \frac{1}{6k^2 A^2} = 0. \tag{11}$$

However, eq, (11) has no real solution, so we conclude that the skier never loses contact with a cosine hill for any values of A and k.

The radius of curvature (3) increases with x up to $x = \pi/2k$ where it is infinite, so the cosine hill is gentler than a cylindrical hill of radius $r = R(0) = 1/k^2 A$, and is sufficiently gentle that the skier never loses contact with a cosine hill.