The Fields in a Box with Resistive Walls

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1 Problem

A uniform electric field is desired throughout a cube of edge a. This can be arranged by constructing a cubical box in which the face at $z = 0$ is a (perfect) conductor at potential 0, the face at $z = a$ is a conductor at potential V_0, and the four remaining faces are made of a poor conductor, say of resistivity ρ_0 Ohms/square.

Suppose, however, the material of the faces between $z = 0$ and a has a nonuniform resistivity which varies as,

$$\rho(z) = \rho_0 \left(1 + \epsilon \sin \frac{2\pi z}{a}\right),$$

where $\epsilon \ll 1$.

An exact solution for the potential inside the box can be given, but is very cumbersome.

Calculate the potential $V(z)$ on the resistive walls, and **estimate** the potential at the center of the cube.

Estimate the maximum ratio of the transverse electric field to the longitudinal field (E_z). Where does this maximum occur?

You may assume the current in the resistive faces flows parallel to the z axis.

2 Solution

We first find the current in the resistive walls, and then calculate the potential as a function of z on the walls. To estimate the potential ϕ as the center of the cube, we use the fact that $\nabla^2\phi = 0$ implies that the potential in the center of a small volume element is the average of the potential over the surrounding surface.

First, the wall current $I = V_0/R$.

The total resistance $R = \int_0^a R(z)dz$, where $R(z) = \rho(z)dz/4a$, since the perimeter of the wall is $4a$. Hence,

$$R = \frac{\rho_0}{4a} \int_0^a \left(1 + \epsilon \sin \frac{2\pi z}{a}\right) dz = \frac{\rho_0}{4a} \left[z - \frac{\epsilon a}{2\pi} \cos \frac{2\pi z}{a}\right]_0^a = \frac{\rho_0}{4}.$$

Then on the wall,

$$V(z) = \int_0^z I dR = \frac{V_0}{R} \frac{\rho_0}{4a} \int_0^z \left(1 + \epsilon \sin \frac{2\pi z}{a}\right) dz = V_0 \left[\frac{z}{a} + \frac{\epsilon}{2\pi} \left(1 - \cos \frac{2\pi z}{a}\right)\right].$$

The average potential on the wall is,

$$\langle\phi\rangle_{\text{wall}} = \frac{V_0}{2} \left(1 + \frac{\epsilon}{\pi}\right).$$

The average potential over the entire surface of the cube is,

$$\langle \phi \rangle = \frac{1}{6}\left[0 + V_0 + 4\frac{V_0}{2}\left(1 + \frac{\epsilon}{\pi}\right)\right] = \frac{V_0}{2}\left(1 + \frac{2\epsilon}{3\pi}\right) \approx \phi_{\rm center}.$$

The electric field in the z direction is $E_z \approx V_0/a$.

The transverse electric field is greatest in the midplane, $z = a/2$, where the wall potential is,

$$\phi_{\rm wall} = \frac{V_0}{2}\left(1 + \frac{2\epsilon}{\pi}\right).$$

The electric field in the x direction is roughly,

$$E_x \approx \frac{\phi_{\rm wall} - \phi_{\rm center}}{a/2} = \frac{4\epsilon V_0}{3\pi a} \approx \frac{4\epsilon}{3\pi} E_z.$$

One might argue that the variation of the potential with x is sine-like, and so the maximum slope is $\pi/2$ times the simplified estimate. Then,

$$E_{x,\rm max} \approx \frac{2\epsilon}{3} E_z.$$

Similarly for E_y.

The transverse field is therefore greatest in the corners of the wall, at the midplane in z, and,

$$\frac{E_{\perp,\rm max}}{E_z} \approx \frac{2\sqrt{2}}{3}\epsilon \approx \epsilon.$$

No doubt, $E_\perp/E_z \approx \epsilon$ could have been guessed at once.