$\nabla \times (\nabla \times \mathbf{E})$ in Spherical Coordinates

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(December 21, 2020)

1 Problem

From Maxwell's first-order differential equations for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ (in Gaussian units),

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \tag{1}$$

where c is the speed of light in vacuum, one obtains second-order wave equations for the field by taking the curl of the curl equations. Thus,

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t}, \tag{2}$$

and using the vector-calculus identity,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \tag{3}$$

and the first Maxwell equation, one arrives at the wave equation for the electric field,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 4\pi \nabla\rho + \frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t}. \tag{4}$$

On a spatial scale large compared to that of the source densities ρ and $\mathbf{J}$, the waves are essentially spherical. This leads to the question: what is $\nabla \times (\nabla \times \mathbf{E})$, and $\nabla^2 \mathbf{E}$, in spherical coordinates? *Consider also cylindrical coordinates.*[1]

2 Solution

The difficulty is that only in rectangular coordinates do all spatial derivatives of coordinate unit vectors vanish. Textbooks often give the expressions for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$ in cylindrical and spherical coordinate systems, but rarely for $\nabla^2 \mathbf{E}$. An exception is [3], pp. 115-117.

First, we note that, in spherical coordinates (r, θ, ϕ),

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta E_\theta) + \frac{1}{r \sin\theta}\frac{\partial E_\phi}{\partial \phi}. \tag{5}$$

[1] Jan. 31, 2022. Dragan Redžić noted that this problem was solved in a general way in [1]. See also [2].

To deduce $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})$ we note that for a scalar field ψ,

$$\boldsymbol{\nabla}\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial\psi}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,. \tag{6}$$

From eqs. (5)-(6), the components of $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})$ are,

$$(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}))_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial r}\right) - \frac{2E_r}{r^2} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial r}\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta)$$
$$+\frac{1}{r\sin\theta}\frac{\partial^2 E_\phi}{\partial r\partial\phi} - \frac{1}{r^2\sin\theta}\frac{\partial E_\phi}{\partial\phi}\,, \tag{7}$$

$$(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}))_\theta = \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial^2 E_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial E_\theta}{\partial\theta} - \frac{E_\theta}{r^2\sin^2\theta}$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial^2 E_\phi}{\partial\theta\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\phi}{\partial\phi}\,, \tag{8}$$

$$(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}))_\phi = \frac{1}{r^3\sin\theta}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial\phi}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\phi}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_\phi}{\partial^2\phi}\,. \tag{9}$$

From p. 116 of [3], the components of $\nabla^2\mathbf{E}$ in spherical coordinates are,

$$(\nabla^2\mathbf{E})_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_r}{\partial\phi^2}$$
$$-\frac{2E_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) - \frac{2}{r^2\sin\theta}\frac{\partial E_\phi}{\partial\phi}\,, \tag{10}$$

$$(\nabla^2\mathbf{E})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_\theta}{\partial\phi^2}$$
$$-\frac{E_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial E_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\phi}{\partial\phi}\,, \tag{11}$$

$$(\nabla^2\mathbf{E})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_\phi}{\partial\phi^2}$$
$$-\frac{E_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial E_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\theta}{\partial\phi}\,. \tag{12}$$

Finally, the components of $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}$ in spherical coordinates are,

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}))_r = -\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_r}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial r\partial\theta}(r\sin\theta E_\theta)$$
$$+\frac{1}{r\sin\theta}\frac{\partial^2 E_\phi}{\partial r\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial E_\phi}{\partial\phi}\,, \tag{13}$$

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}))_\theta = \frac{1}{r}\frac{\partial^2 E_r}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\theta}{\partial r}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_\theta}{\partial\phi^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\phi}{\partial\phi}\right), \tag{14}$$

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}))_\phi = \frac{1}{r\sin\theta}\frac{\partial^2 E_r}{\partial r\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2 E_\theta}{\partial\theta\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\theta}{\partial\phi}$$
$$+\frac{E_\phi}{r^2\sin^2\theta} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\phi}{\partial r}\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\phi}{\partial\theta}\right). \tag{15}$$

3 Examples

3.1 Point Charge

The electric field of a point charge q at the origin is,

$$\mathbf{E} = \frac{q}{r^2}\,\hat{\mathbf{r}}\,. \tag{16}$$

Away from the origin, the wave equations (2) and (4) become,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\nabla^2 \mathbf{E} = 0. \tag{17}$$

The only nontrivial component of $\nabla^2\mathbf{E}$ is, from eq. (10),

$$(\nabla^2\mathbf{E})_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial r}\right) - \frac{2E_r}{r^2} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2q}{r}\right) - \frac{2q}{r^4} = \frac{2q}{r^4} - \frac{2q}{r^4} = 0. \tag{18}$$

In eq. (13) there is no nontrivial component of $(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r$ for a point charge, *i.e.*,

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r = 0. \tag{19}$$

3.2 Electric Dipole

3.2.1 $\mathbf{p} = p\,\hat{\mathbf{z}}$

We first consider an electric dipole aligned along the z-axis, for which the electric field is,

$$\mathbf{E} = \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}}{r^3} = \frac{p(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{r^3}\,, \qquad E_r = \frac{2p\cos\theta}{r^3}\,, \qquad E_\theta = \frac{p\sin\theta}{r^3}\,. \tag{20}$$

Again, away from the origin, the wave equations are given by eq. (17). From eqs. (10)-(11),

$$\begin{aligned}
(\nabla^2\mathbf{E})_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right) - \frac{2E_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) \\
&= -\frac{6p\cos\theta}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) - \frac{2p}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^3}\right) - \frac{4p\cos\theta}{r^5} - \frac{2p}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^3}\right) \\
&= \frac{12p\cos\theta}{r^5} - \frac{4p\cos\theta}{r^5} - \frac{4p\cos\theta}{r^5} - \frac{4p\cos\theta}{r^5} = 0,
\end{aligned} \tag{21}$$

$$\begin{aligned}
(\nabla^2\mathbf{E})_\theta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\theta}{\partial r}\right) + \frac{1}{r^5\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\theta}\right) - \frac{E_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial E_r}{\partial\theta} \\
&= -\frac{3p\sin\theta}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) + \frac{p}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) - \frac{p}{r^5\sin\theta} - \frac{4\sin\theta}{r^5} \\
&= \frac{6p\sin\theta}{r^5} + \frac{p(\cos^2\theta - \sin^2\theta)}{r^5\sin\theta} - \frac{p}{r^5\sin\theta} - \frac{4\sin\theta}{r^5} = 0.
\end{aligned} \tag{22}$$

From eqs. (13)-(14),

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r = -\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial E_r}{\partial\theta} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial^2}{\partial r \partial\theta} (r \sin\theta E_\theta)$$

$$= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\frac{2p \sin^2\theta}{r^3} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial^2}{\partial r \partial\theta} \left(\frac{p \sin^2\theta}{r^2} \right) = \frac{4p\cos\theta}{r^5} - \frac{4p\cos\theta}{r^5} = 0, \tag{23}$$

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_\theta = \frac{1}{r} \frac{\partial^2 E_r}{\partial r \partial\theta} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial E_\theta}{\partial r} \right) = \frac{6p\sin\theta}{r^5} - \frac{6p\sin\theta}{r^5} = 0. \tag{24}$$

3.2.2 $\mathbf{p} = p\,\hat{\mathbf{x}}$

For an example in which E_ϕ is nonzero, we consider a (static) point electric dipole at the origin, aligned with the x-axis. The electric scalar potential is then,

$$V = \frac{p\cos\theta_x}{r^2} = \frac{p\sin\theta\cos\phi}{r^2}. \tag{25}$$

The electric field components are, from $\mathbf{E} = -\boldsymbol{\nabla} V$,

$$E_r = \frac{2p\sin\theta\cos\phi}{r^3} \qquad E_\theta = -\frac{p\cos\theta\cos\phi}{r^3}, \qquad E_\phi = \frac{p\sin\phi}{r^3}. \tag{26}$$

Away from the origin, we expect that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = 0$.

From eqs. (13)-(15),

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r = -\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial E_r}{\partial\theta} \right) - \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 E_r}{\partial\phi^2} + \frac{1}{r^2 \sin\theta} \frac{\partial^2}{\partial r \partial\theta} (r \sin\theta E_\theta)$$

$$+ \frac{1}{r\sin\theta} \frac{\partial^2 E_\phi}{\partial r \partial\phi} + \frac{1}{r^2 \sin\theta} \frac{\partial E_\phi}{\partial\phi}$$

$$= -\frac{2p(1 - 2\sin^2\theta)\cos\phi}{r^5 \sin\theta} + \frac{2p\cos\phi}{r^5\sin\theta} + \frac{2p(1 - 2\sin^2\theta)\cos\phi}{r^5\sin\theta} - \frac{3p\cos\phi}{r^5\sin\theta} + \frac{p\cos\phi}{r^5\sin\theta}$$

$$= 0, \tag{27}$$

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_\theta = \frac{1}{r} \frac{\partial^2 E_r}{\partial r \partial\theta} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial E_\theta}{\partial r} \right) - \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 E_\theta}{\partial\phi^2} + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial E_\phi}{\partial\phi} \right)$$

$$= -\frac{6p\cos\theta\cos\phi}{r^5} + \frac{6p\cos\theta\cos\phi}{r^5} - \frac{p\cos\theta\cos\phi}{r^5\sin^2\theta} + \frac{p\cos\theta\cos\phi}{r^5\sin^2\theta} = 0, \tag{28}$$

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_\phi = \frac{1}{r\sin\theta} \frac{\partial^2 E_r}{\partial r \partial\phi} + \frac{1}{r^2 \sin\theta} \frac{\partial^2 E_\theta}{\partial\theta \partial\phi} - \frac{\cos\theta}{r^2 \sin^2\theta} \frac{\partial E_\theta}{\partial\phi}$$

$$+ \frac{E_\phi}{r^2 \sin^2\theta} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial E_\phi}{\partial r} \right) - \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial E_\phi}{\partial\theta} \right)$$

$$= \frac{6p\sin\phi}{r^5} - \frac{p\sin\phi}{r^5} - \frac{p\sin\phi}{r^5\sin^2\theta} + \frac{p\sin\phi}{r^5} + \frac{p\sin\phi}{r^5\sin^2\theta} - \frac{6p\sin\phi}{r^5} = 0. \tag{29}$$

3.3 Hertzian Electric Dipole Radiation

For an example of a time-dependent electric field, we consider an ideal, oscillating, Hertzian (point) electric dipole $\mathbf{p} = p_0\, e^{-i\omega t}\,\hat{\mathbf{z}}$ at the origin, for which the electric field can be written as the real part of (see, for example, sec. 9.2 of [6]),

$$
\begin{aligned}
\mathbf{E} &= k^2 p_0 (\hat{\mathbf{r}} \times \hat{\mathbf{z}}) \times \hat{\mathbf{r}}\, \frac{e^{i(kr-\omega t)}}{r} + p_0 [3(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{z}}] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)} \\
&= -k^2 p_0 \sin\theta\, \hat{\boldsymbol{\theta}}\, \frac{e^{i(kr-\omega t)}}{r} + p_0 (2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}\,, \qquad (30)
\end{aligned}
$$

where $k = \omega/c$. Away from the origin, the wave equation (2) is,

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = k^2 \mathbf{E} \\
&= -k^4 p_0 \sin\theta\, \hat{\boldsymbol{\theta}}\, \frac{e^{i(kr-\omega t)}}{r} + k^2 p_0 (2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}\,. \qquad (31)
\end{aligned}
$$

From eqs. (13)-(14),

$$
\begin{aligned}
(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r &= -\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial E_r}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial r\partial\theta}(r\sin\theta E_\theta) \\
&= \frac{2p_0}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}\right) \\
&\quad + \frac{p_0}{r^2\sin\theta}\frac{\partial^2}{\partial r\partial\theta}\left(\sin^2\theta\left(-k^2 + \frac{1}{r^2} - \frac{ik}{r}\right) e^{i(kr-\omega t)}\right) \\
&= \frac{4p_0\cos\theta}{r^2}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)} + \frac{2p_0\cos\theta}{r^2}\frac{\partial}{\partial r}\left(-k^2 + \frac{1}{r^2} - \frac{ik}{r}\right) e^{i(kr-\omega t)} \\
&= \frac{4p_0\cos\theta}{r^2}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)} \\
&\quad + \frac{2p_0\cos\theta}{r^2}\left(-\frac{2}{r^3} + \frac{ik}{r^2}\right) e^{i(kr-\omega t)} + \frac{2p_0\cos\theta}{r^2}\left(-ik^3 + \frac{ik}{r^2} + \frac{k^2}{r}\right) e^{i(kr-\omega t)} \\
&= \frac{2p_0\cos\theta}{r^2}\left(\frac{2}{r^3} - \frac{2ik}{r^2} - \frac{2}{r^3} + \frac{ik}{r^2} - ik^3 + \frac{ik}{r^2} + \frac{k^2}{r}\right) e^{i(kr-\omega t)} \\
&= 2k^2 p_0\cos\theta\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}, \qquad (32) \\
(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_\theta &= \frac{1}{r}\frac{\partial^2 E_r}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\theta}{\partial r}\right) \\
&= -\frac{2p_0\sin\theta}{r}\frac{\partial}{\partial r}\left(\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}\right) \\
&\quad - \frac{p_0\sin\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(-\frac{k^2}{r} + \frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}\right) \\
&= -\frac{2p_0\sin\theta}{r}\left(-\frac{3}{r^4} + \frac{2ik}{r^3} + \frac{ik}{r^3} + \frac{k^2}{r^2}\right) e^{i(kr-\omega t)}
\end{aligned}
$$

$$-\frac{p_0 \sin\theta}{r^2} \frac{\partial}{\partial r} \left(r^2 \left(\frac{k^2}{r^2} - \frac{3}{r^4} + \frac{2ik}{r^3} - \frac{ik^3}{r} + \frac{ik}{r^3} + \frac{k^2}{r^2} \right) e^{i(kr-\omega t)} \right)$$
$$= -p_0 \sin\theta \left(-\frac{6}{r^5} + \frac{6ik}{r^4} + \frac{2k^2}{r^3} \right) e^{i(kr-\omega t)}$$
$$-\frac{p_0 \sin\theta}{r^2} \frac{\partial}{\partial r} \left(\left(2k^2 - \frac{3}{r^2} + \frac{3ik}{r} - ik^3 r \right) e^{i(kr-\omega t)} \right)$$
$$= -p_0 \sin\theta \left(-\frac{6}{r^5} + \frac{6ik}{r^4} + \frac{2k^2}{r^3} \right) e^{i(kr-\omega t)}$$
$$-\frac{p_0 \sin\theta}{r^2} \left(\frac{6}{r^3} - \frac{6ik}{r^2} + ik^3 - \frac{6k^2}{r} + k^4 r \right) e^{i(kr-\omega t)}$$
$$= -p_0 \sin\theta \left(\frac{k^4}{r} + \frac{ik^3}{r^2} - \frac{k^2}{r^3} \right) e^{i(kr-\omega t)}, \tag{33}$$

which agrees with eq. (31).

A Appendix: Cylindrical Coordinates

First, we note that, in cylindrical coordinates (r, ϕ, z),

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial r}(r E_r) + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} . \tag{34}$$

To deduce $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E})$ we note that for a scalar field ψ,

$$\boldsymbol{\nabla} \psi = \hat{\mathbf{r}} \frac{\partial \psi}{\partial r} + \frac{\hat{\boldsymbol{\phi}}}{r} \frac{\partial \psi}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial \psi}{\partial z} . \tag{35}$$

From eqs. (34)-(35), the components of $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E})$ are,

$$(\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}))_r = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_r}{\partial r} \right) - \frac{E_r}{r^2} - \frac{1}{r^2} \frac{\partial E_\phi}{\partial \phi} + \frac{1}{r} \frac{\partial^2 E_\phi}{\partial r \partial \phi} + \frac{\partial^2 E_z}{\partial r \partial z} , \tag{36}$$
$$(\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}))_\phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r \frac{\partial E_r}{\partial \phi} \right) + \frac{1}{r^2} \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{1}{r} \frac{\partial^2 E_z}{\partial \phi \partial z} , \tag{37}$$
$$(\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}))_z = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r \partial E_r}{\partial z} \right) + \frac{1}{r} \frac{\partial^2 E_\phi}{\partial \phi \partial z} + \frac{\partial^2 E_z}{\partial z^2} . \tag{38}$$

From p. 116 of [3], the components of $\nabla^2 \mathbf{E}$ in cylindrical coordinates are,

$$(\nabla^2 \mathbf{E})_r = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_r}{\partial \phi^2} + \frac{\partial^2 E_r}{\partial z^2} - \frac{E_r}{r^2} - \frac{2}{r^2} \frac{\partial E_\phi}{\partial \phi} , \tag{39}$$
$$(\nabla^2 \mathbf{E})_\phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial z^2} - \frac{E_\phi}{r^2} + \frac{2}{r^2} \frac{\partial E_r}{\partial \phi} , \tag{40}$$
$$(\nabla^2 \mathbf{E})_z = \nabla^2 E_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} . \tag{41}$$

Finally, the components of $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ in cylindrical coordinates are,

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r = -\frac{1}{r^2}\frac{\partial^2 E_r}{\partial \phi^2} - \frac{\partial^2 E_r}{\partial z^2} + \frac{1}{r^2}\frac{\partial E_\phi}{\partial \phi} + \frac{1}{r}\frac{\partial^2 E_\phi}{\partial r \partial \phi} + \frac{\partial^2 E_z}{\partial r \partial z}\,, \tag{42}$$

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial E_r}{\partial \phi}\right) - \frac{2}{r^2}\frac{\partial E_r}{\partial \phi} + \frac{E_\phi}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_\phi}{\partial r}\right) - \frac{\partial^2 E_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial^2 E_z}{\partial \phi \partial z}\,, \tag{43}$$

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_z = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial E_r}{\partial z}\right) + \frac{1}{r}\frac{\partial^2 E_\phi}{\partial \phi \partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2}\,. \tag{44}$$

A.1 Examples

A.1.1 Line Charge

The electric field of a line charge λ per unit length along the z-axis is,

$$\mathbf{E} = \frac{\lambda}{r}\,\hat{\mathbf{r}}\,. \tag{45}$$

Away from the origin, the wave equations (2) and (4) become,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\nabla^2 \mathbf{E} = 0. \tag{46}$$

The only nontrivial component of $\nabla^2 \mathbf{E}$ is, from eq. (39),

$$(\nabla^2 \mathbf{E})_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_r}{\partial r}\right) - \frac{E_r}{r^2} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\lambda}{r}\right) - \frac{\lambda}{r^3} = \frac{\lambda}{r^3} - \frac{\lambda}{r^3} = 0. \tag{47}$$

In eq. (42) there is no nontrivial component of $(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r$ for a line charge, *i.e.*,

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r = 0. \tag{48}$$

A.1.2 Conducting Cylinder in a Uniform, Transverse, External Electric Field

The scalar potential for this case can be written as (see, for example, p. 52 of [5]),

$$V(r > a, \phi) = -E_0\left(r - \frac{a^2}{r}\right)\cos\phi, \tag{49}$$

for a conducting cylinder of radius a in an external field $\mathbf{E}_0 = E_0\,\hat{\mathbf{x}}$. The electric-field components are,

$$E_r = -\frac{\partial V}{\partial r} = E_0\left(1 + \frac{a^2}{r^2}\right)\cos\phi, \qquad E_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = -E_0\left(1 - \frac{a^2}{r^2}\right)\sin\phi\,. \tag{50}$$

Outside the cylinder, the wave equations are given by eq. (17). From eqs. (39)-(40),

$$(\nabla^2 \mathbf{E})_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_r}{\partial \phi^2} - \frac{E_r}{r^2} - \frac{2}{r^2}\frac{\partial E_\phi}{\partial \phi}$$

$$= -\frac{E_0}{r}\frac{\partial}{\partial r}\left(\frac{2a}{r^2}\right)\cos\phi - E_0\left(\frac{1}{r^2}+\frac{a^2}{r^4}\right)\cos\phi - E_0\left(\frac{1}{r^2}+\frac{a^2}{r^4}\right)\cos\phi + E_0\left(\frac{2}{r^2}-\frac{2a^2}{r^4}\right)\cos\phi$$

$$= E_0\frac{4a^2}{r^4}\cos\phi - E_0\frac{4a^2}{r^4}\cos\phi = 0, \tag{51}$$

$$(\nabla^2\mathbf{E})_\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_\phi}{\partial\phi^2} - \frac{E_\phi}{r^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial\phi}$$

$$= \frac{E_0}{r}\frac{\partial}{\partial r}\left(\frac{2a^2}{r^2}\right)\sin\phi + E_0\left(\frac{1}{r^2}-\frac{a^2}{r^4}\right)\sin\phi + E_0\left(\frac{1}{r^2}-\frac{a^2}{r^4}\right)\sin\phi - E_0\left(\frac{2}{r^2}+\frac{2a^2}{r^4}\right)\sin\phi$$

$$-E_0\left(\frac{4a^2}{r^4}\right)\sin\phi + E_0\left(\frac{4a^2}{r^4}\right)\sin\phi = 0. \tag{52}$$

From eqs. (42)-(43),

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}))_r = -\frac{1}{r^2}\frac{\partial^2 E_r}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial E_\phi}{\partial\phi} + \frac{1}{r}\frac{\partial^2 E_\phi}{\partial r\partial\phi}$$

$$= E_0\left(\frac{1}{r^2}+\frac{a^2}{r^4}\cos\phi\right) - E_0\left(\frac{1}{r^2}-\frac{a^2}{r^4}\cos\phi\right) - E_0\frac{2a^2}{r^4}\cos\phi = 0, \tag{53}$$

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}))_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial E_r}{\partial\phi}\right) - \frac{2}{r^2}\frac{\partial E_r}{\partial\phi} + \frac{E_\phi}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_\phi}{\partial r}\right)$$

$$= -E_0\left(\frac{1}{r^2}-\frac{a^2}{r^4}\right)\sin\phi + E_0\left(\frac{2}{r^2}+\frac{2a^2}{r^4}\right)\sin\phi - E_0\left(\frac{1}{r}-\frac{a^2}{r^2}\right)\sin\phi + \frac{E_0}{r}\frac{\partial}{\partial r}\left(\frac{2a^2}{r^2}\right)\sin\phi$$

$$= E_0\frac{4a^2}{r^4}\sin\phi - E_0\frac{4a^2}{r^4}\sin\phi = 0. \tag{54}$$

A.1.3 TM Modes of a Circular Waveguide

For an example in which the electric field depends on z, we consider the transverse magnetic (TM) modes of a (vacuum) circular waveguide of radius a. The z-axis is also that of the guide.

The electric field components (for $r < a$) can be written as (see [4]),

$$E_r = \frac{i\,E_0 k_{z,mn} a}{u_{mn}} J'_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}, \tag{55}$$

$$E_\phi = \mp\frac{mE_0 k_{z,mn} a^2}{u_{mn}^2 r} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}, \tag{56}$$

$$E_z = E_0 J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}, \tag{57}$$

where m and n are integers, $m \geq 0$, $n \geq 1$, the u_{mn} are the zeros of the ordinary Bessel function of order m,

$$J_m(u_{mn}) = 0, \tag{58}$$

and,

$$k_0 = \frac{\omega}{c}, \qquad k_{mn}^2 = \frac{u_{mn}^2}{a^2}, \qquad k_{z,mn}^2 = k_0^2 - k_{mn}^2 = \frac{\omega^2}{c^2} - \frac{u_{mn}^2}{a^2}. \tag{59}$$

Note that $k_{z,mn}$ and not k_{mn} or k_0 is the propagation constant, and that the minimum angular frequency of a propagating wave is,

$$\omega_{mn,\min} = \frac{u_{mn}\, c}{a}\,. \tag{60}$$

Inside the guide, the wave equation (2) is,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\omega^2}{c^2}\mathbf{E}. \tag{61}$$

From eqs. (42)-(44),

$$\begin{aligned}
(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_r &= -\frac{1}{r^2}\frac{\partial^2 E_r}{\partial \phi^2} - \frac{\partial^2 E_r}{\partial z^2} + \frac{1}{r^2}\frac{\partial E_\phi}{\partial \phi} + \frac{1}{r}\frac{\partial^2 E_\phi}{\partial r \partial \phi} + \frac{\partial^2 E_z}{\partial r \partial z} \\
&= \frac{im^2 E_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad + \frac{iE_0 k_{z,mn}^3 a}{u_{mn}} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad - \frac{im^2 E_0 k_{z,mn} a^2}{u_{mn}^2 r^3} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad + \frac{imE_0 k_{z,mn} a^2}{u_{mn}^2 r^3} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} - \frac{imE_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad + \frac{iE_0 k_{z,mn} u_{mn}}{a} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&= \left(k_{z,mn}^2 + \frac{u_{mn}^2}{a^2}\right) \frac{iE_0 k_{z,mn} a}{u_{mn}} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} = \frac{\omega^2}{c^2} E_r,
\end{aligned} \tag{62}$$

$$\begin{aligned}
(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}))_\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial E_r}{\partial \phi}\right) - \frac{2}{r^2}\frac{\partial E_r}{\partial \phi} + \frac{E_\phi}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_\phi}{\partial r}\right) - \frac{\partial^2 E_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial^2 E_z}{\partial \phi \partial z} \\
&= \mp\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{mE_0 k_{z,mn} a}{u_{mn}} r J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)}\right) \\
&\quad \pm \frac{2mE_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad \mp \frac{mE_0 k_{z,mn} a^2}{u_{mn}^2 r^3} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad \pm \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{mE_0 k_{z,mn} a^2 r}{u_{mn}^2}\frac{\partial}{\partial r}\left(\frac{1}{r}J_m\left(\frac{r}{a}u_{mn}\right)\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)}\right) \\
&\quad \mp \frac{mE_0 k_{z,mn}^3 a^2}{u_{mn}^2 r} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad \mp \frac{mE_0 k_{z,mn}}{r} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&= \mp\frac{mE_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \mp \frac{mE_0 k_{z,mn}}{r} J_m''\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)} \\
&\quad \pm \frac{2mE_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn} z - \omega t)}
\end{aligned}$$

$$
\begin{aligned}
&\mp \frac{mE_0 k_{z,mn} a^2}{u_{mn}^2 r^3} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{mE_0 k_{z,mn} a^2}{r u_{mn}^2} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}\right) \\
&\pm \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{mE_0 k_{z,mn} a}{u_{mn}} J_m'\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}\right) \\
&\mp \frac{mE_0 k_{z,mn}^3 a^2}{u_{mn}^2 r} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{mE_0 k_{z,mn}}{r} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
= &\pm \frac{mE_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \mp \frac{mE_0 k_{z,mn}}{r} J_m''\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{mE_0 k_{z,mn} a^2}{u_{mn}^2 r^3} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\pm \frac{mE_0 k_{z,mn} a^2}{r^3 u_{mn}^2} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{mE_0 k_{z,mn} a}{r^2 u_{mn}} J_m'\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\pm \frac{mE_0 k_{z,mn}}{r} J_m''\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{mE_0 k_{z,mn}^3 a^2}{u_{mn}^2 r} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{mE_0 k_{z,mn}}{r} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
= &\mp \frac{mE_0 k_{z,mn} a^2}{u_{mn}^2 r}\left(k_{z,mn}^2 + \frac{u_{mn}^2}{a^2}\right) J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} = \frac{\omega^2}{c^2}E_\phi,
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}))_z = &\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial E_r}{\partial z}\right) + \frac{1}{r}\frac{\partial^2 E_\phi}{\partial\phi\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 E_z}{\partial\phi^2} \\
= &\pm \frac{mE_0 k_{z,mn} a}{u_{mn} r^2} J_m'\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\mp \frac{mE_0 k_{z,mn} a^2}{u_{mn}^2 r}\left(\frac{1}{r^2} + k_{z,mn}^2 + \frac{u_{mn}^2}{a^2}\right) J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
= &-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{E_0 k_{z,mn}^2 a}{u_{mn}} r J_m'\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}\right) \\
&+ \frac{m^2 E_0 k_{z,mn}^2 a^2}{u_{mn}^2 r^2} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&- \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{E_0 u_{mn}}{a} r J_m'\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}\right) \\
&+ \frac{m^2 E_0}{r^2} J_m\left(\frac{r}{a}u_{mn}\right)\, e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{E_0 k_{z,mn}^2 a}{u_{mn} r} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} - E_0 k_{z,mn}^2 J_m''\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\qquad\qquad + \frac{m^2 E_0 k_{z,mn}^2 a^2}{u_{mn}^2 r^2} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\quad - \frac{E_0 u_{mn}}{ar} J_m'\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} - \frac{E_0 u_{mn}^2}{a^2} J_m''\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&\qquad\qquad + \frac{m^2 E_0}{r^2} J_m\left(\frac{r}{a}u_{mn}\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&= \frac{\omega^2}{c^2} E_0 \left(-\frac{a}{r u_{mn}} J_m'\left(\frac{r}{a}u_{mn}\right) - J_m''\left(\frac{r}{a}u_{mn}\right) + \frac{m^2 a^2}{r^2 u_{mn}^2} J_m\left(\frac{r}{a}u_{mn}\right)\right) e^{\pm im\phi}\, e^{i(k_{z,mn}z-\omega t)} \\
&= \frac{\omega^2}{c^2} E_z,
\end{aligned}
\tag{64}
$$

noting that the second to last line in eq. (64) contains the Bessel equation satisfied by $J_m(r u_{mn}/a)$ (eq. (23) of [4]),

$$
J_m''\left(\frac{r}{a}u_{mn}\right) + \frac{a}{r u_{mn}} J_m'\left(\frac{r}{a}u_{mn}\right) + \left(1 - \frac{m^2 a^2}{r^2 u_{mn}^2}\right) J_m\left(\frac{r}{a}u_{mn}\right) = 0. \tag{65}
$$

Acknowledgment

Thanks to Dragan Redžić for spotting several typos in earlier versions of this note.

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